Short Takes 331

Roots on the complex plane

Roots on the complex plane

- Suppose we need to solve

$$\underbrace{z^2 - i}_{f(z)} = 0 \qquad , \quad z = x + iy$$

$$f(z) = g_1(x,y) + i\, g_2(x,y)$$

$$z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$$

$$\Rightarrow \begin{cases} g_1(x,y) = x^2 - y^2 \\ g_2(x,y) = 2xy - 1 \end{cases}$$

- We want to use 2D Newton-Raphson...

$$g_1(x,y) = 0 \qquad\qquad x, y = ?$$
$$g_2(x,y) = 0$$

$$\vec{F} = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix} \qquad\qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} \quad , \quad \frac{\partial g_1}{\partial x} = 2x \quad , \quad \frac{\partial g_1}{\partial y} = -2y$$

$$\frac{\partial g_2}{\partial x} = 2y \quad , \quad \frac{\partial g_2}{\partial y} = 2x$$

We need to solve

$$J \, \Delta\vec{x} = -\vec{F} \quad , \quad \text{to calculate } \vec{x}_1 = \vec{x}_0 + \Delta\vec{x}$$

Cramer's rule

$$\Delta x = \frac{1}{\det J} \cdot \begin{vmatrix} -g_1 & \frac{\partial g_1}{\partial y} \\ -g_2 & \frac{\partial g_2}{\partial y} \end{vmatrix} = \frac{1}{\det J}\left(g_2 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_2}{\partial y} \right)$$

Note: all entries, and $\det J$, are functions of x, y

$$\Delta y = \frac{1}{\det J} \cdot \begin{vmatrix} \frac{\partial g_1}{\partial x} & -g_1 \\ \frac{\partial g_2}{\partial x} & -g_2 \end{vmatrix} = \frac{1}{\det J}\left(g_1 \frac{\partial g_2}{\partial x} - g_2 \frac{\partial g_1}{\partial x} \right)$$

$$\det J = \frac{\partial g_1}{\partial x}\frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial x}$$

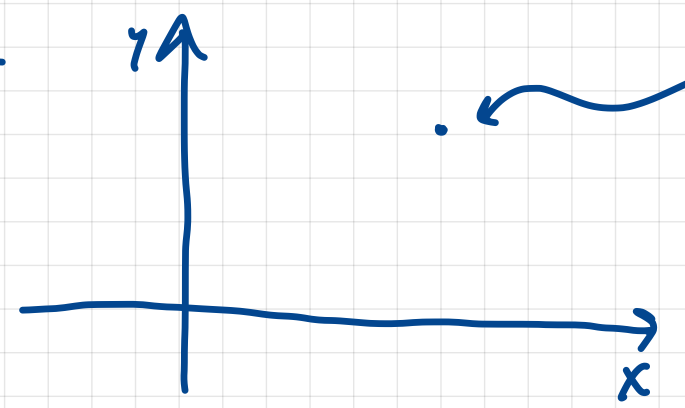
$$= 4(x^2 + y^2)$$

Update

$$\begin{cases} x_1 = x_0 + \Delta x \\ y_1 = y_0 + \Delta y \end{cases},$$

Use $|f(x,y)| = \left[(g_1(x,y))^2 + (g_2(x,y))^2 \right]^{1/2}$ for stopping criterion.

Plot



give a color to this starting point indicating which root it converges to and how fast (to achieve a given precision).