Short Takes 331

Roots in 2D

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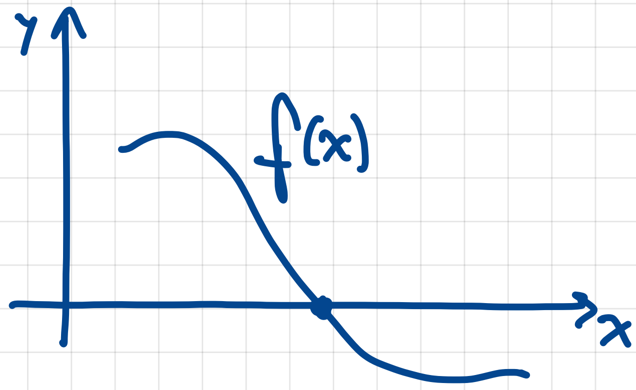
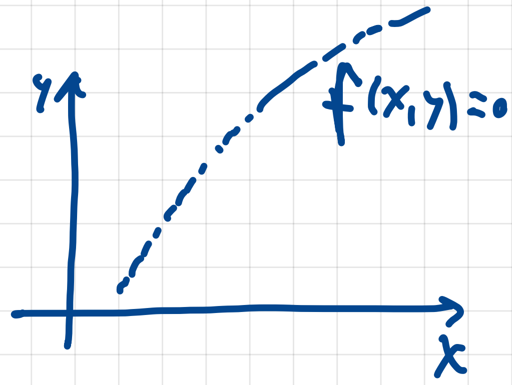
So far, $f(x) = 0$

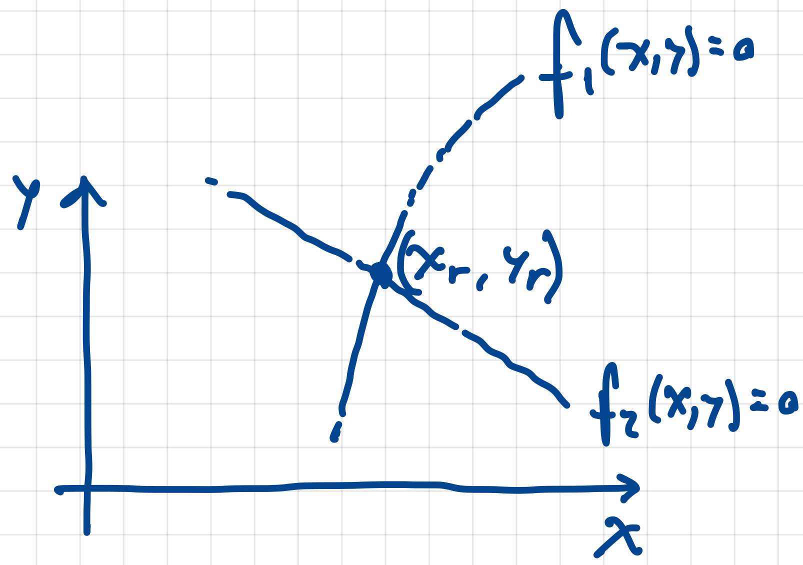


What if...

- $f(x,y) = 0 \longrightarrow$ a whole family of solutions (x,y)



- $\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases} \longrightarrow$ a solution (x_r, y_r)



- **How to proceed?**

Suppose we are close to the root.
We are standing at (x,y) and we want to find Δx, Δy such that

$$\begin{cases} x_r = x + \Delta x \\ y_r = y + \Delta y \end{cases}$$

, i.e. the step $(\Delta x, \Delta y)$ lands on the root.

Using Taylor...

$$\begin{cases} f_1(x+\Delta x, y+\Delta y) = f_1(x,y) + \dfrac{\partial f_1}{\partial x}(x,y)\cdot\Delta x + \dfrac{\partial f_1}{\partial y}(x,y)\cdot\Delta y + \dots \\ f_2(x+\Delta x, y+\Delta y) = f_2(x,y) + \dfrac{\partial f_2}{\partial x}(x,y)\cdot\Delta x + \dfrac{\partial f_2}{\partial y}(x,y)\cdot\Delta y + \dots \end{cases}$$

rewrite more conveniently...

$$\vec{F}(\vec{x} + \Delta\vec{x}) = \vec{F}(\vec{x}) + J\cdot\Delta\vec{x} + \dots$$

$$\vec{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \Delta\vec{x} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

"Jacobian matrix"

If $\Delta\vec{x}$ lands on the root, then

$$\vec{F}(\vec{x}+\Delta\vec{x}) = \vec{F}(\vec{x}) + J \cdot \Delta\vec{x} = 0$$

$$\Rightarrow \quad J \cdot \Delta\vec{x} = -\vec{F}(\vec{x}) \quad \Rightarrow \quad \boxed{\Delta\vec{x} = -J^{-1} \cdot \vec{F}(\vec{x})}$$

In the Newton-Raphson method, our job is to:

- Pick $\vec{x}_0 = (x_0, y_0)$
- Calculate $\vec{F}(\vec{x}_0)$
- Calculate $J(\vec{x}_0)$
- Calculate $\Delta\vec{x} = -J^{-1}(\vec{x}_0) \cdot \vec{F}(\vec{x}_0)$ ($\longrightarrow$ use a linear solver)
- Update $\vec{x}_1 = \vec{x}_0 + \Delta\vec{x}$
- Return to the top with $\vec{x}_0 \longrightarrow \vec{x}_1$.

- Where can things go wrong?
  - $\longrightarrow$ $\det J = 0$ or J is approximately singular
- Generalizable to arbitrary dimension.