Short Takes 331

Roots via Newton-Raphson

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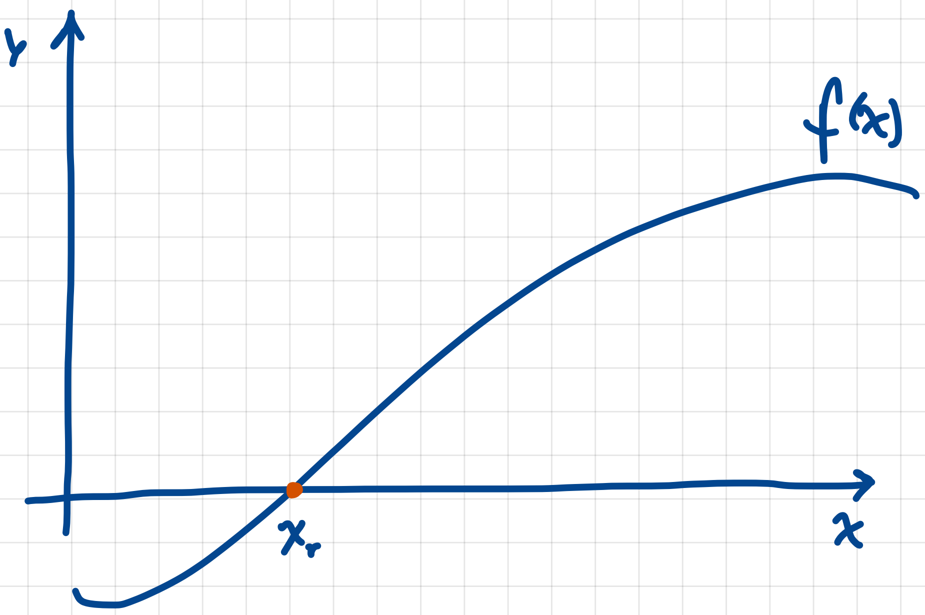
We want to calculate the root position x_r

$$f(x_r) = 0$$



Idea: close enough to the root x_r, the function is approximately linear (hopefully!).

Then,

$$f(x) = \underbrace{f(x_r)}_{=0} + f'(x_r)(x - x_r) + \ldots$$

If that approximation holds, then it must be that

$$f(x) = f'(x_r)(x - x_r)$$

$$\Rightarrow \quad \frac{f(x)}{f'(x_r)} = x - x_r \quad \Rightarrow \quad x_r = x - \frac{f(x)}{f'(x_r)} \simeq x - \frac{f(x)}{f'(x)}$$

$$\boxed{x_r \simeq x - \frac{f(x)}{f'(x)}}$$

Newton-Raphson step

1. Pick a point x_0 close to x_r.
2. Use the formula above to calculate an approximation for x_r. Call it x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

3. Return to step 2 using x_1 instead of x_0.

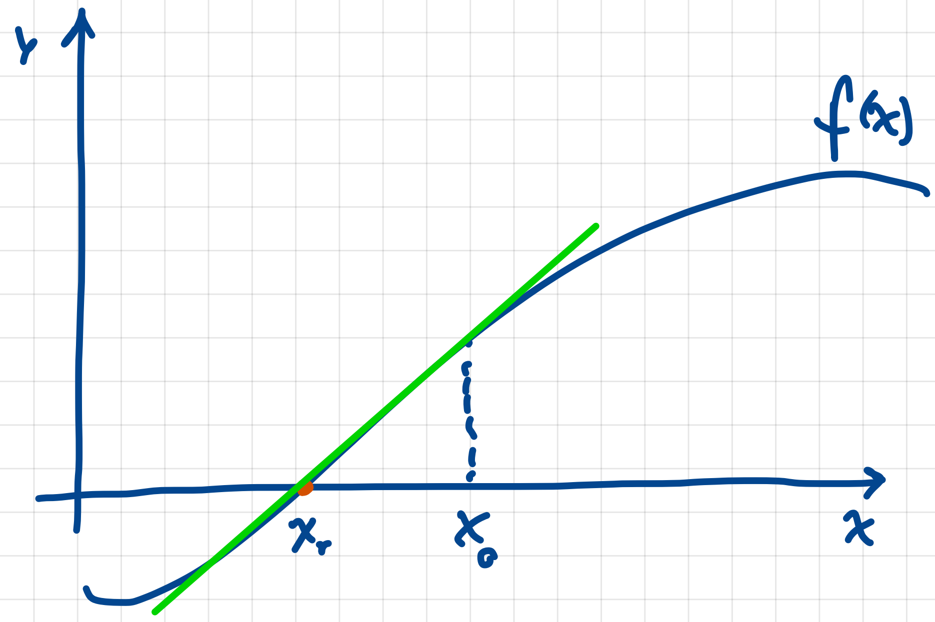
- Another way to see the derivation & geometry...
  - Pick a point x_0 close to the root and calculate the tangent line equation:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

  - set $y=0$, i.e. extrapolate to the x axis and calculate the x location of the x-axis crossing:

$$0 = f(x_0) + f'(x_0)(x - x_0)$$

$$\Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$



  - Call this new point x_1 and go back to the beginning.

- Caveats
  - This is a method for refining roots. You must have x_0 close to x_r.
  - Beware of points where $f'(x) = 0$!

What could you do if you knew $f'(x_r) = 0$ but $f''(x_r) \neq 0$?