Short Takes 331

Green’s function for diffusion in 1D Part 2

Green's function for diffusion in 1D. Part 2.

- **Last time**: inhomogeneous diffusion eq.

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = j(x,t) \qquad\qquad f(x,0) \quad \text{i.c.}$$

inhomogeneity
(driving the system; usually given)

$$\rightarrow Df = j \,, \qquad D = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

$$f = D^{-1} j$$

$$f(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,t;\underline{x',t'})\, j(\underline{x',t'})\, dx'\, dt'$$

"Green's function"

$$DG(x,t;x',t') = \delta(x-x')\,\delta(t-t')$$

$$\boxed{G(x,t;x',t') = \Theta(t-t')\,\frac{1}{2\pi}\sqrt{\frac{\pi}{t-t'}}\; e^{-\frac{(x-x')^2}{4(t-t')}}}$$

- Suppose

$$j(x,t) = e^{-x^2} e^{-t^2}$$

Then,

$$f(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} \frac{1}{2\pi}\sqrt{\frac{\pi}{t-t'}}\; e^{-\frac{(x-x')^2}{4(t-t')}}\, e^{-x'^2}\, e^{-t'^2}\, dx'\, dt',$$

but what about the initial condition ...?

• At $t=0$,

$$\longrightarrow \underbrace{\int_{-\infty}^{0}\int_{-\infty}^{\infty} \frac{1}{2\pi}\sqrt{\frac{\pi}{-t'}}\, e^{-\frac{(x-x')^2}{4(-t')}}\, e^{-x'^2}\, e^{-t'^2}\, dx'\, dt'}$$

" $g(x) \neq$ given initial condition $f(x,0)$

• How do we enforce

$f(x,t) \longrightarrow f(x,0)$ (given), once we've accounted for $j(x,t)$?

$$\rightarrow \boxed{f = D^{-1} j + h}$$

solution of homogeneous problem $Dh = 0$, which fixes the initial condition.

Notice that we still have

$$Df = DD^{-1}j + Dh = j \quad ✓$$

Now, at $t=0$,

$$f(x,t) \longrightarrow g(x) + h(x,0) \overset{!}{=} f(x,0)$$

$\Rightarrow h(x,t) =$ solution of $Dh = 0$

with i.c. $h(x,0) = f(x,0) - g(x)$

• But then this means we didn't really have the right D^{-1}; we had the one for $f(x,0) = g(x)$.

(Notice we ignored the initial condition entirely when building G.)

This is ok - it's just a reflection of the fact that the kernel of D is nontrivial:

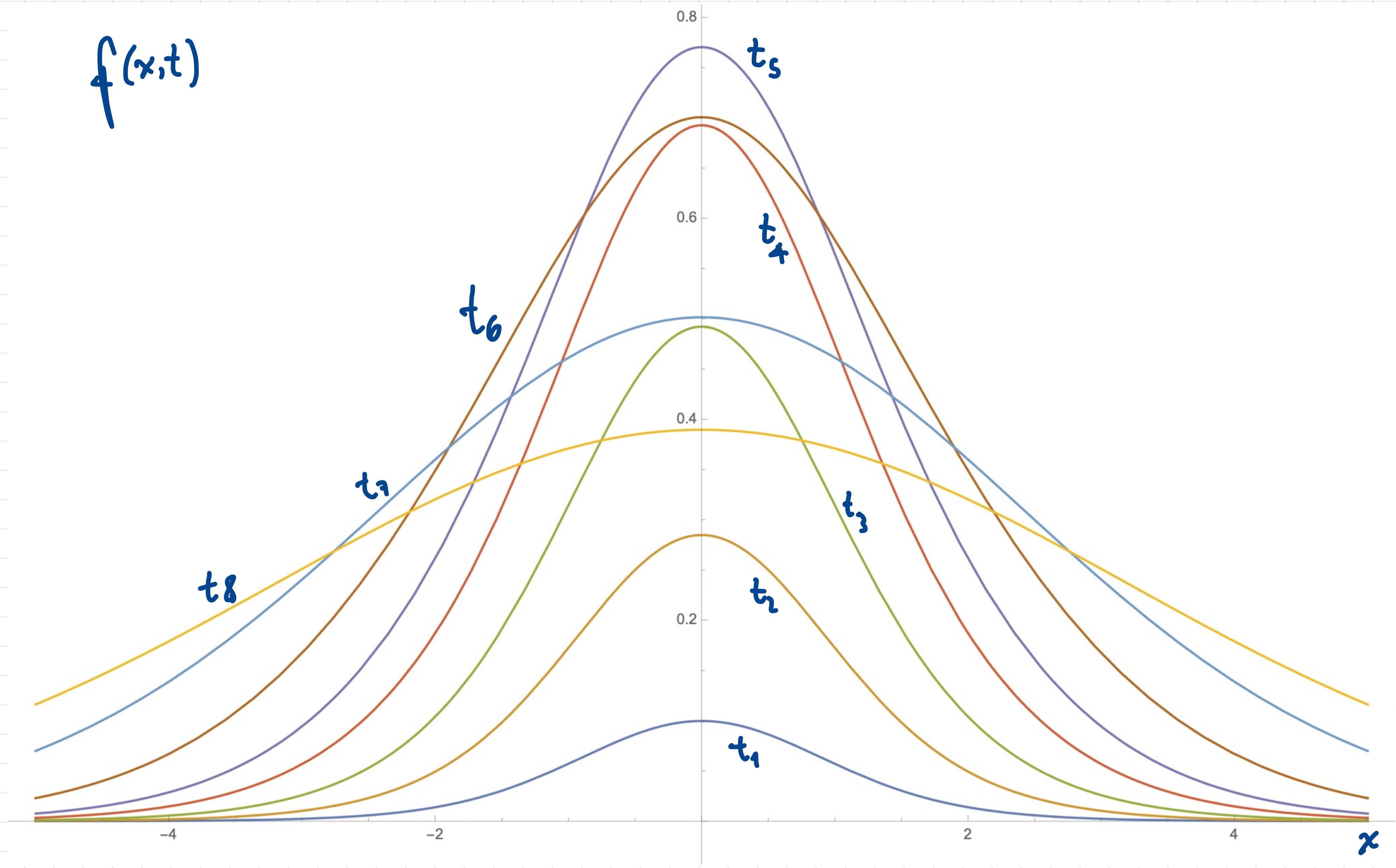
$Df = 0$ has nontrivial solutions which depend on the initial and boundary conditions.

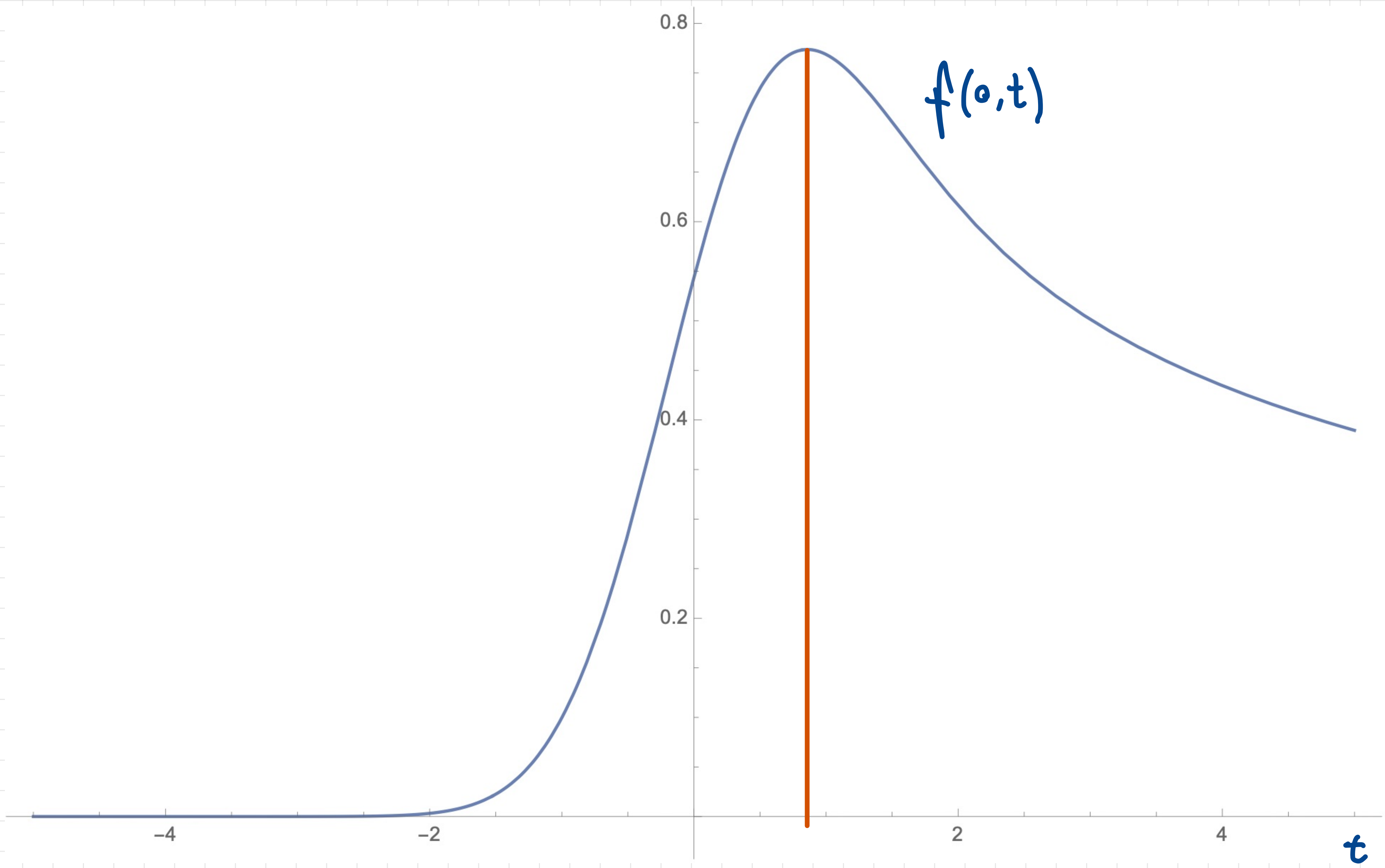
- For simplicity, take i.c. at $t \to -\infty$

$$f(x,t) \xrightarrow[t \to -\infty]{} 0$$

, which is reproduced with $h \equiv 0$ in our case

$$j(x,t) = e^{-x^2} e^{-t^2}$$





- Note delay in $f(x,t)$ vs. $j(x,t)$
- We used contour integration to obtain $G(x,t;x',t')$ → more on that in future videos!
- $f = D^{-1} j + h$

  $Dh = 0$