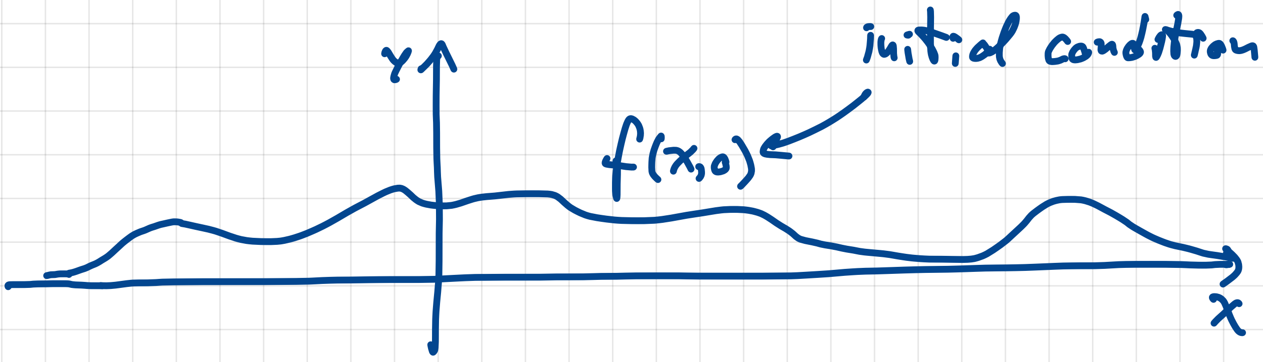
Short Takes 331

Green's function for diffusion in 1D
Part 1

Green's function for diffusion in 1D

- **Homogeneous problem:**

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = 0$$



boundaries at $\pm\infty$.

- **Inhomogeneous problem:**

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = j(x,t)$$

inhomogeneity
(driving the system; usually given)

Recall... $\quad Df = j \quad , \quad D = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$

$$\Rightarrow \quad f = D^{-1} j$$

$$f(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,t;\underline{x',t'})\, j(\underline{x',t'})\, dx'\, dt'$$

"Green's function"

$$\boxed{DG(x,t;x',t') = \delta(x-x')\,\delta(t-t')}$$

Recall completeness relation of Fourier modes...

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk$$

and similarly...

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\, d\omega$$

- Now we propose a Fourier decomposition for G:

$$G(x,t;x',t') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{G}(k,\omega)\, e^{ik(x-x')}\, e^{-i\omega(t-t')}\, dk\, d\omega$$

Then,

$$DG = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{G}(k,\omega)\,(-i\omega+k^2)\, e^{ik(x-x')}\, e^{-i\omega(t-t')}\, dk\, d\omega$$

$$\downarrow$$

$$D = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

$$= \delta(x-x')\,\delta(t-t')$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-x')}\, e^{-i\omega(t-t')}\, dk\, d\omega$$

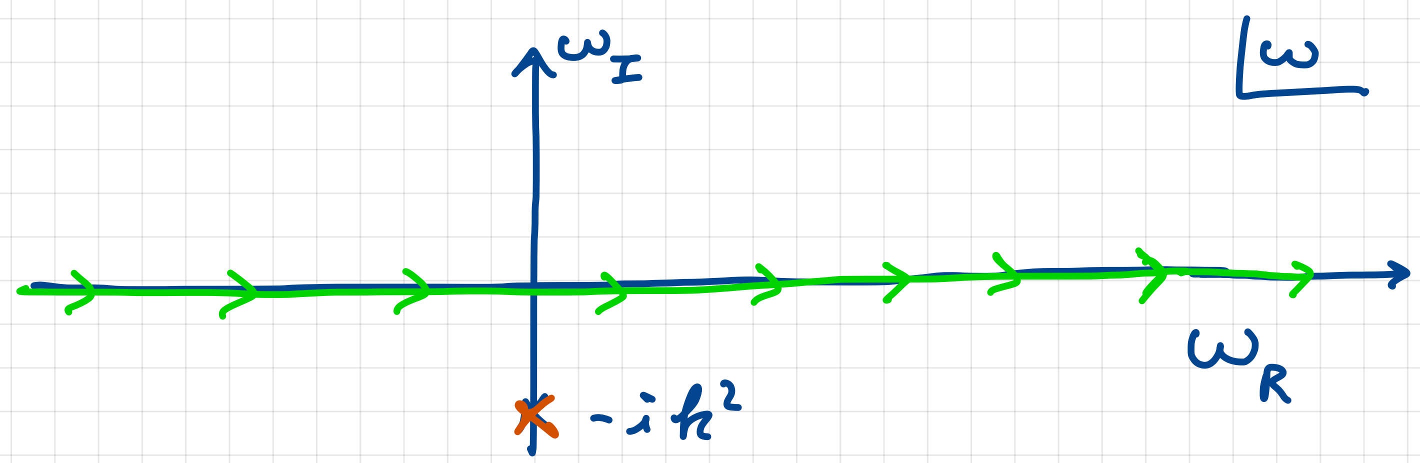
$$\Rightarrow \boxed{\tilde{G}(k,\omega) = \left(\frac{1}{2\pi}\right)^2 \cdot (-i\omega+k^2)^{-1}}$$

Therefore

$$G(x,t;x',t') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi}\right)^2 (-i\omega+k^2)^{-1}}_{\tilde{G}(k,\omega)}\, e^{ik(x-x')}\, e^{-i\omega(t-t')}\, dk\, d\omega$$

- How to do the integrals over k, ω? $\rightarrow$ First ω!

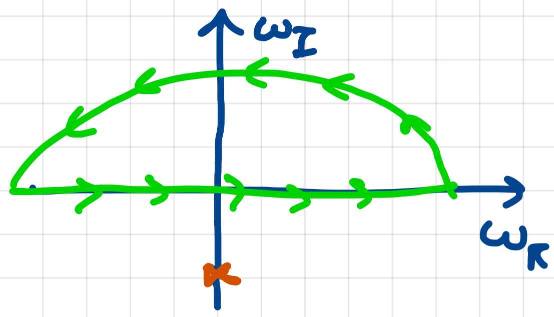
$(-i\omega + k^2)^{-1}$: simple pole at $\omega = -ik^2$



- $t < t'$: $G(x,t;x',t') = 0$ $\rightarrow$ close contour around top half

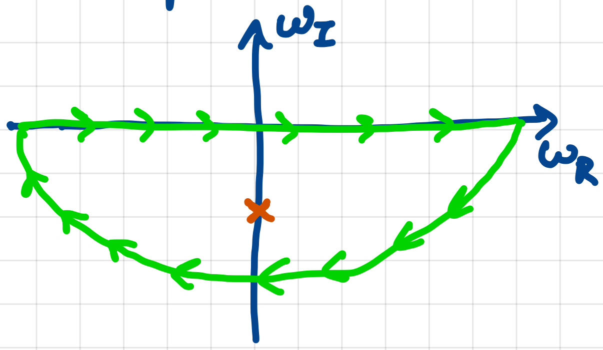
"Nothing happens before a signal is picked up."

$\rightarrow$ "Retarded Green's function"



note: counterclockwise!

- $t \geq t'$: close contour around bottom half



note: clockwise!

contour integration using residue theorem.

$$G(x,t;x',t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} e^{-k^2(t-t')} dk \quad , \quad t > t'$$

complete the square

$\longrightarrow$

$\Delta x = x - x'$

$\Delta t = t - t'$

$$k^2\Delta t - ik\Delta x = \Delta t\left[k^2 - 2k\,\frac{i\,\Delta x}{2\Delta t} + \left(i\,\frac{\Delta x}{2\Delta t}\right)^2 - \left(i\,\frac{\Delta x}{2\Delta t}\right)^2\right]$$

$$= \Delta t\left[k - \frac{i\,\Delta x}{2\Delta t}\right]^2 + \frac{(\Delta x)^2}{4\Delta t}$$

$$G(x,t;x',t') = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} e^{-(t-t')\left[k - \frac{i(x-x')}{2(t-t')}\right]^2} dk \; e^{-\frac{(x-x')^2}{4(t-t')}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{t-t'}}\; e^{-\frac{(x-x')^2}{4(t-t')}} \qquad t \geq t'$$

$$G(x,t;x',t') = \Theta(t-t')\,\frac{1}{2\pi}\sqrt{\frac{\pi}{t-t'}}\; e^{-\frac{(x-x')^2}{4(t-t')}}$$

Heaviside theta function $= \begin{cases} 0 & t < t' \\ 1 & t \geq t' \end{cases}$

Application ... next time!

What happened to the initial condition $f(x,0)$? Stay tuned!