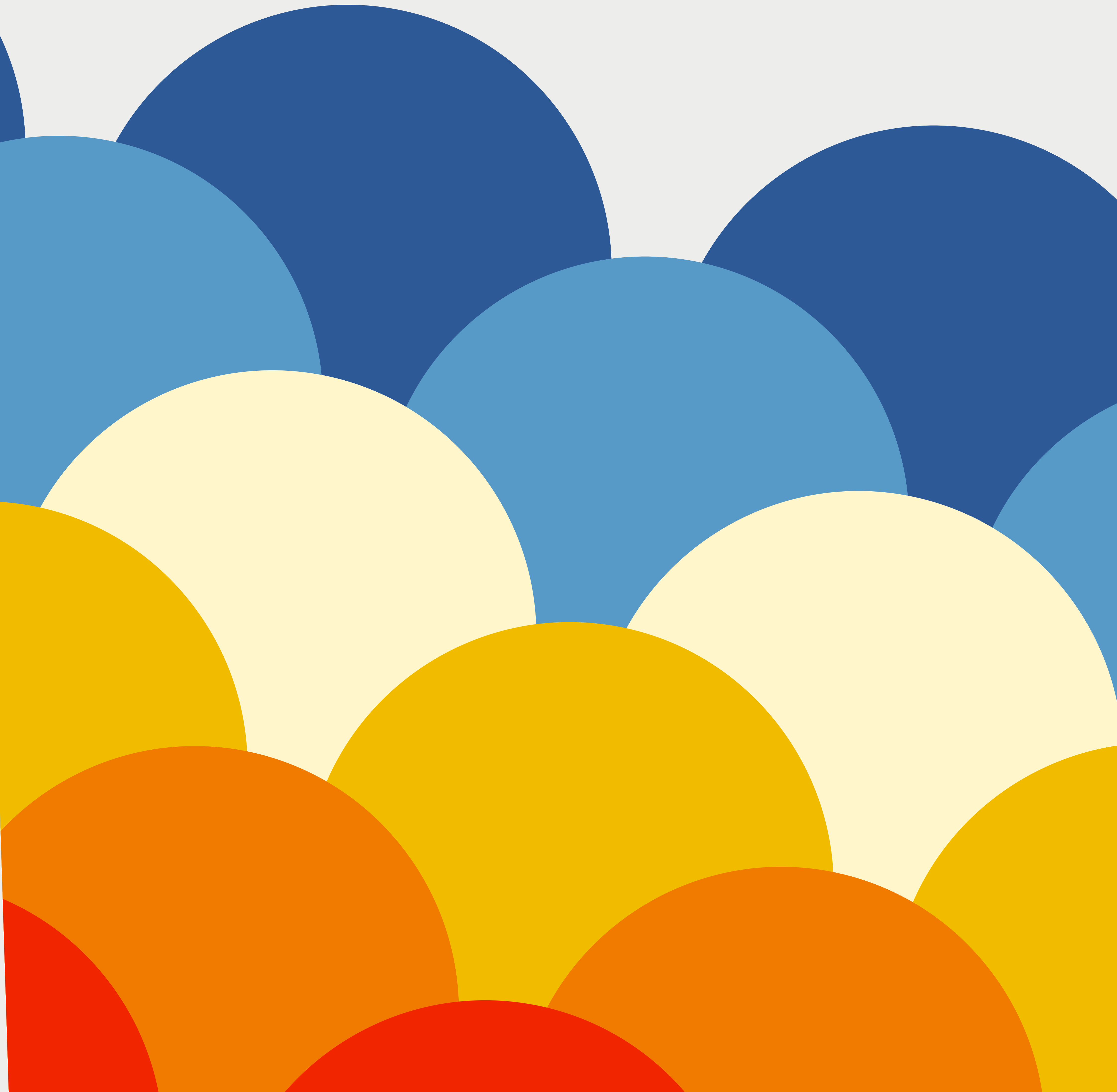
Short Takes 331

Another Fourier way to diffusion and waves



Another Fourier way for diffusion & waves

Recall our setup on the whole real line ...

- Diffusion: $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ (take $D=1$) + i.c. $f(x,0)$

- Waves: $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$ (take $c=1$) + i.c. $f(x,0)$, $f'(x,0)$

Propose

$$f(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(k,\omega)\, e^{ikx}\, e^{-i\omega t}\, dk\, d\omega$$

$\Rightarrow$ Diffusion:

$$-i\omega \tilde{f}(k,\omega) = (ik)^2 \tilde{f}(k,\omega) = -k^2 \tilde{f}(k,\omega)$$

$$(-i\omega + k^2)\tilde{f}(k,\omega) = 0$$

$$\Rightarrow \boxed{\tilde{f}(k,\omega) = \tilde{f}_0(k)\,\delta(\omega + ik^2)}$$

It must be this way, otherwise $\tilde{f}(k,\omega) \equiv 0$.

Waves:

$$(-i\omega)^2 \tilde{f}(k,\omega) = -k^2 \tilde{f}(k,\omega)$$

$$(\omega^2 - k^2)\tilde{f}(k,\omega) = 0$$

$$\Rightarrow \boxed{\tilde{f}(k,\omega) = \tilde{f}_1(k)\,\delta(\omega - k) + \tilde{f}_2(k)\,\delta(\omega + k)}$$

Then,

Diffusion:

$$f(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}_0(k)\, \delta(\omega + ik^2)\, e^{ikx} e^{-i\omega t}\, dk\, d\omega$$

$$= \int_{-\infty}^{\infty} \tilde{f}_0(k)\, e^{-k^2 t} e^{ikx}\, dk$$

Waves:

$$f(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\tilde{f}_1(k)\delta(\omega - k) + \tilde{f}_2(k)\delta(\omega + k)\right] e^{ikx} e^{-i\omega t}\, dk\, d\omega$$

$$= \int_{-\infty}^{\infty} \tilde{f}_1(k)\, e^{ik(x-t)}\, dk + \int_{-\infty}^{\infty} \tilde{f}_2(k)\, e^{ik(x+t)}\, dk$$

- This approach can be generalized to arbitrary dimension:

$$e^{ikx} e^{-i\omega t} \longrightarrow e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t}$$

$$\vec{k}\cdot\vec{r} = k_1 x + k_2 y + k_3 z \qquad (3D)$$

- This type of approach to solving diff. eqs. will usually work if your eqs. are <u>linear</u>.

Moreover, you should pay attention to the geometry of the region of interest & the boundary conditions.

(Our FT approach worked because the region was all $\mathbb{R}$.)