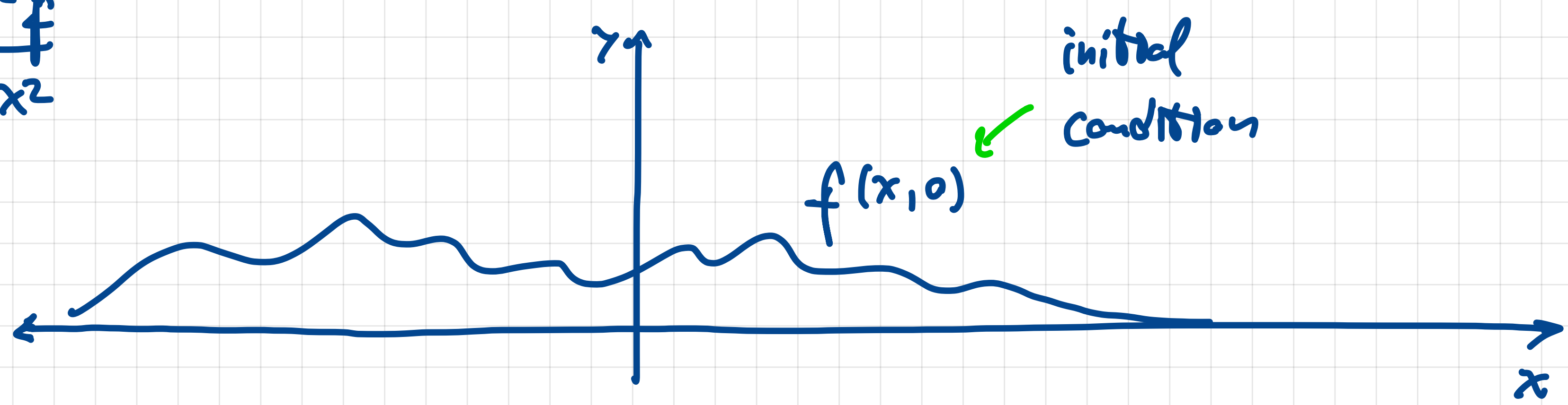
Short Takes 331

The Fourier transform Part 4: Waves!

The Fourier transform. Part 4: Waves!

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$



→ How does this initial waveform evolve as a function of time?

→ Propose a Fourier decomposition (aka "superposition")

$$f(x,t) = \int_{-\infty}^{\infty} \tilde{f}(k,t)\, e^{ikx}\, dk$$

time-dep Fourier transform

helps solve the spatial part

→ Using our eqn.:

$$\int_{-\infty}^{\infty} \frac{1}{c^2}\frac{\partial^2 \tilde{f}}{\partial t^2}(k,t)\, e^{ikx}\, dk = \int_{-\infty}^{\infty} \tilde{f}(k,t)(ik)^2\, e^{ikx}\, dk$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \tilde{f}}{\partial t^2}(k,t) = (ik)^2 \tilde{f}(k,t) = -k^2 \tilde{f}(k,t)$$

$$\Rightarrow \quad \tilde{f}(k,t) = A_k e^{ickt} + B_k e^{-ickt}$$

<u>Note</u>: $\tilde{f}(k,0) = A_k + B_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,0)\, e^{-ikx}\, dx$

$\tilde{f}'(k,0) = ick(A_k - B_k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x,0)\, e^{-ikx}\, dx$

Finally,

$$\boxed{f(x,t) = \int_{-\infty}^{\infty} \left[A_k e^{ickt} + B_k e^{-ickt}\right] e^{ikx}\, dk}$$

Example

1.
- $f(x,0) = \delta(x-a)$
- $f'(x,0) = 0 \implies A_k = B_k$

$$\Rightarrow 2A_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,0)\, e^{-ikx}\, dx = \frac{1}{2\pi} e^{-ika}$$

$$\Rightarrow f(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(e^{ickt} + e^{-ickt}\right) e^{ik(x-a)}\, dk$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{ik(x-a+ct)}\, dk + \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{ik(x-a-ct)}\, dk$$

$$= \underline{\frac{1}{2}\left[\delta(x-a+ct) + \delta(x-a-ct)\right]}$$

2.
- $f(x,0) = g(x) = \int_{-\infty}^{\infty} g(y)\,\delta(x-y)\, dy$
- $f'(x,0) = 0$

$$\Rightarrow 2A_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\, e^{-iky}\, dy$$

$$\Rightarrow f(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} g(y)\left[\delta(x-y+ct) + \delta(x-y-ct)\right] dy$$

$$= \underline{\frac{1}{2}\left[g(x+ct) + g(x-ct)\right]}$$