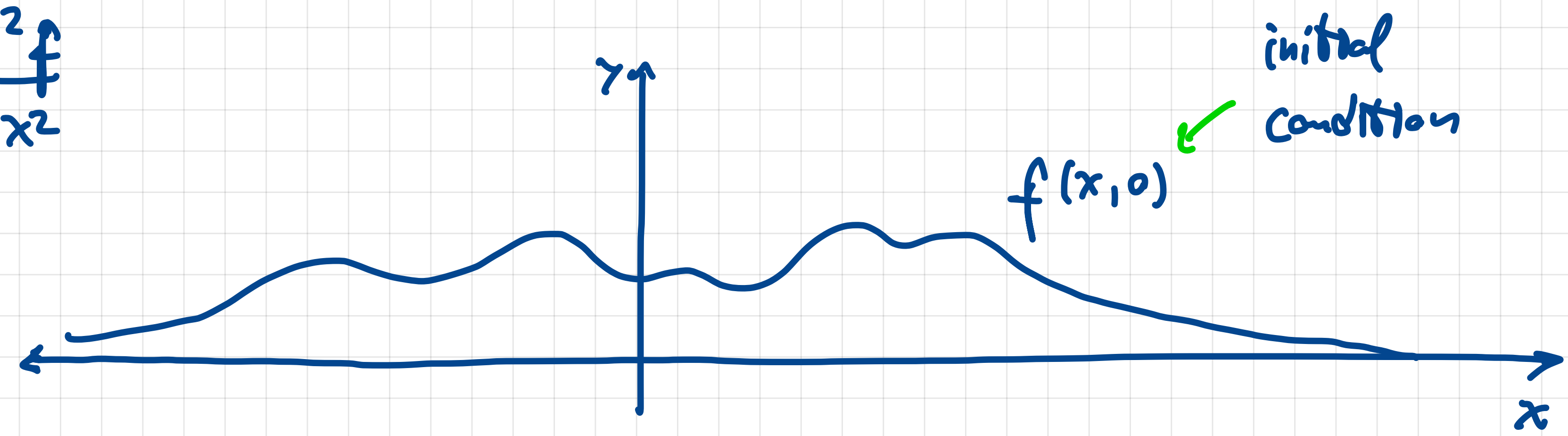
Short Takes 331

The Fourier transform Part 3: Diffusion!

The Fourier transform. Part 3: Diffusion in free space

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$



→ How does this profile (of temperature, etc) evolve as a function of time?

We know

$$\frac{\partial^2}{\partial x^2}\left(e^{ikx}\right) = (ik)^2 e^{ikx}$$

→ e^{ikx} is an eigenfunction of $\frac{\partial^2}{\partial x^2}$ for any k.

→ Propose a Fourier decomposition (aka "superposition")

$$f(x,t) = \int_{-\infty}^{\infty} \tilde{f}(k,t)\, e^{ikx}\, dk$$

time-dep Fourier transform

helps solve the spatial part

Using our eqn.:

$$\int_{-\infty}^{\infty} \frac{\partial \tilde{f}(k,t)}{\partial t}\, e^{ikx}\, dk = \int_{-\infty}^{\infty} \tilde{f}(k,t)(ik)^2\, e^{ikx}\, dk$$

$$\Rightarrow \quad \frac{\partial \tilde{f}}{\partial t}(k,t) = (ik)^2 \tilde{f}(k,t) = -k^2 \tilde{f}(k,t)$$

$$\Rightarrow \quad \tilde{f}(k,t) = \tilde{f}(k,0)\, e^{-k^2 t}$$

known:

$$f(x,0) = \int_{-\infty}^{\infty} \tilde{f}(k,0)\, e^{ikx}\, dk$$

$$\Rightarrow \quad \tilde{f}(k,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,0)\, e^{-ikx}\, dx$$

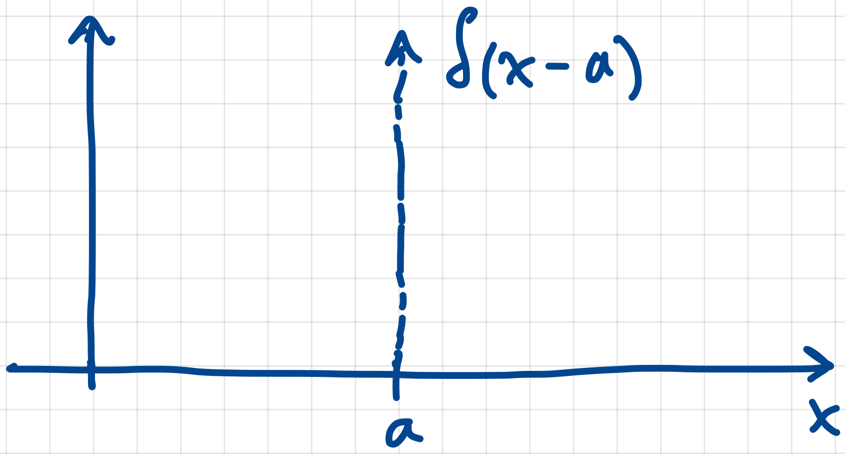
Finally,

$$\boxed{f(x,t) = \int_{-\infty}^{\infty} \tilde{f}(k,0)\, e^{-k^2 t}\, e^{ikx}\, dk}$$

- Examples

- $f(x,0) = \delta(x-a)$

$$\Rightarrow \quad \tilde{f}(k,0) = \frac{1}{2\pi}\, e^{-ika}$$



Then, $$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t}\, e^{ik(x-a)}\, dk$$

→ How do we do this integral? Complete the square!

$$k^2 t - ik(x-a) = t\left[k^2 - 2k\,\frac{i(x-a)}{2t} + \left(\frac{i(x-a)}{2t}\right)^2 - \left(\frac{i(x-a)}{2t}\right)^2\right]$$

$$= t\left[k - \frac{i(x-a)}{2t}\right]^2 + \frac{(x-a)^2}{4t}$$

$$\Rightarrow \quad f(x,t) = \frac{1}{2\pi}\cdot \int_{-\infty}^{\infty} e^{-t\left[k - \frac{i(x-a)}{2t}\right]^2} dk\;\; e^{-\frac{(x-a)^2}{4t}}$$

$$\Rightarrow \boxed{f(x,t) = \frac{1}{2\pi}\cdot\sqrt{\frac{\pi}{t}}\; e^{-\frac{(x-a)^2}{4t}}}$$

our $\delta(x-a)$ is diffusing!

- At $t=0$, $f(x,0) = \delta(x-a)$
- At $t>0$, $f(x,t)$ is a Gaussian with:
  - max height $\frac{1}{2\pi}\sqrt{\frac{\pi}{t}}$ at $x=a$

    decreases with time!

  - width $2\sqrt{t}$

    increases with time!

  - Note: $\displaystyle\int_{-\infty}^{\infty} f(x,t)\,dx = \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\,2\sqrt{t.\pi} = 1$

    time-independent!

- $f(x,0) = e^{-x^2}$

  → What is $\tilde{f}(k,0)$?

  → What is $f(x,t)$?

  → How does the solution behave with t and x?