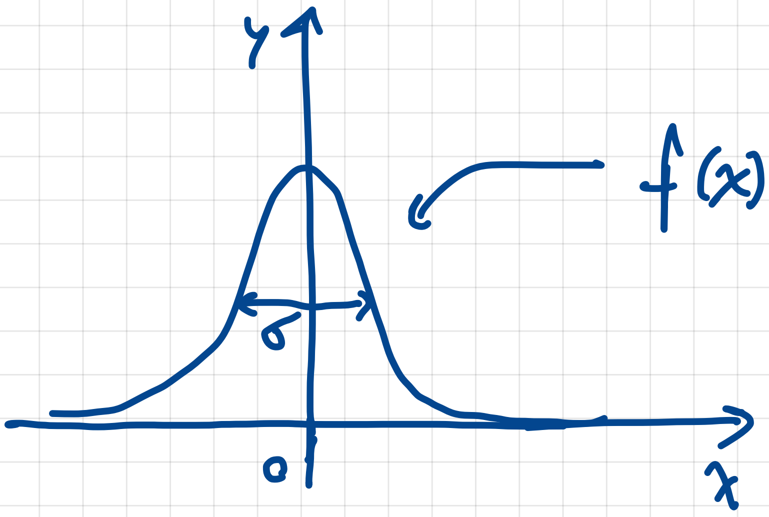
Short Takes 331

The Fourier transform Part 2

The Fourier transform, Part 2.

- What is the Fourier transform of a Gaussian curve?



$$f(x) = A e^{-\left(\frac{x}{\sigma}\right)^2}$$

whatever appears dividing x is the width of the Gaussian.

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2} - ikx}\, dx$$

"complete the square"

$$\frac{x^2}{\sigma^2} + ikx = \frac{1}{\sigma^2}\left(x^2 + ik\sigma^2 x\right)$$

$$= \frac{1}{\sigma^2}\left[x^2 + 2x\, ik\frac{\sigma^2}{2} + \left(\frac{ik\sigma^2}{2}\right)^2 - \left(\frac{ik\sigma^2}{2}\right)^2\right]$$

$$= \frac{1}{\sigma^2}\left[x + \frac{ik\sigma^2}{2}\right]^2 + \frac{k^2\sigma^2}{4}$$

$$= \frac{A}{2\pi}\underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{\sigma^2}\left[x + \frac{ik\sigma^2}{2}\right]^2} dx}_{=\ \sigma\sqrt{\pi}}\; e^{-\frac{k^2\sigma^2}{4}}$$

$$y = x + \frac{ik\sigma^2}{2}$$
$$dy = dx$$

$$= \frac{A}{2\pi}\cdot\sigma\sqrt{\pi}\; e^{-\left(\frac{k}{2/\sigma}\right)^2}$$

The FT of a Gaussian is another Gaussian.

The width of this k-space Gaussian is $\frac{2}{\sigma}$!

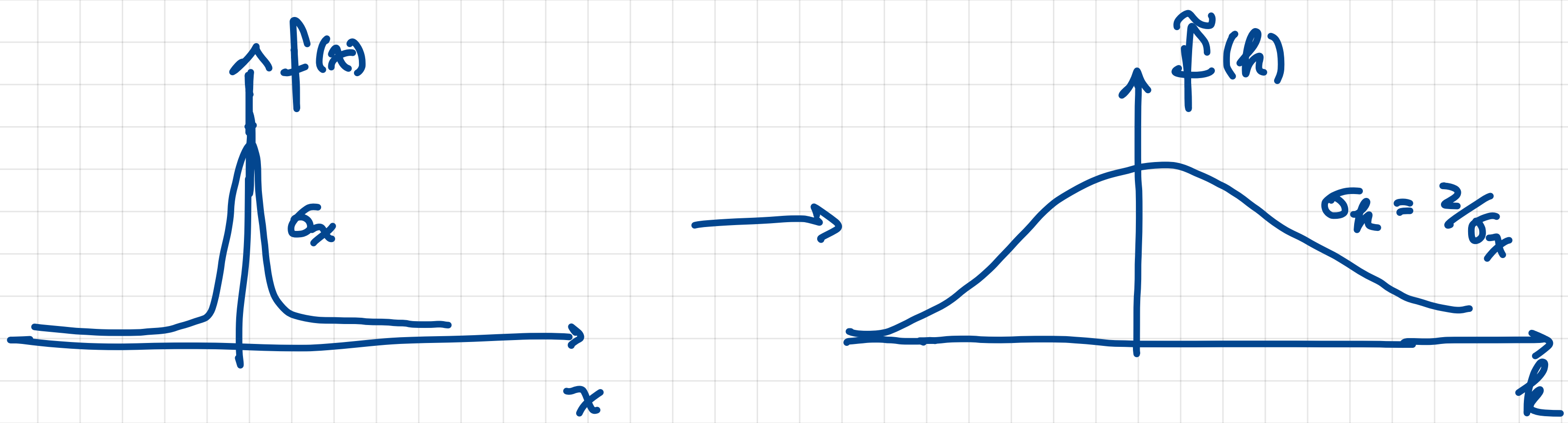
- Let's call σ_x the width in x space and
  σ_k the width in k space.

Then,

$$\sigma_x \cdot \sigma_k = \sigma \cdot \frac{2}{\sigma} = 2$$

→ Uncertainty principle!

- In QM, coordinates and momenta are dual descriptions connected by the Fourier transform.
  Precision in x means σ_x is small, but then σ_k is large!
  Viceversa, if σ_k is small (well-defined momentum), σ_x is large.



- This property is true regardless of QM and applies to any waves you may want to analyze with a FT.
  A signal that is localized in one space will be de-localized in the other space.

- Exercise: Do the integral $\int_{-\infty}^{\infty} e^{-x^2} e^{-ikx}\, dx$ by expanding the exponential as a power series and using our knowledge of Gaussian integrals from previous videos.
