Short Takes 331

The Fourier transform Part 1

The Fourier transform. Part 1.

Fourier transforms come in different "flavors"

Fourier series	**Discrete Fourier transform**	**Fourier transform**
Finite, continuous interval	Finite, discrete interval	Entire real line
Series: converges under certain conditions	**Finite sum**: always converges	**Integral**: converges under certain conditions
Result: discrete set of coefficients, typically infinite	**Result**: finite, discrete set of coefficients	**Result**: function of "conjugate variable" k (usually frequency or wave number)

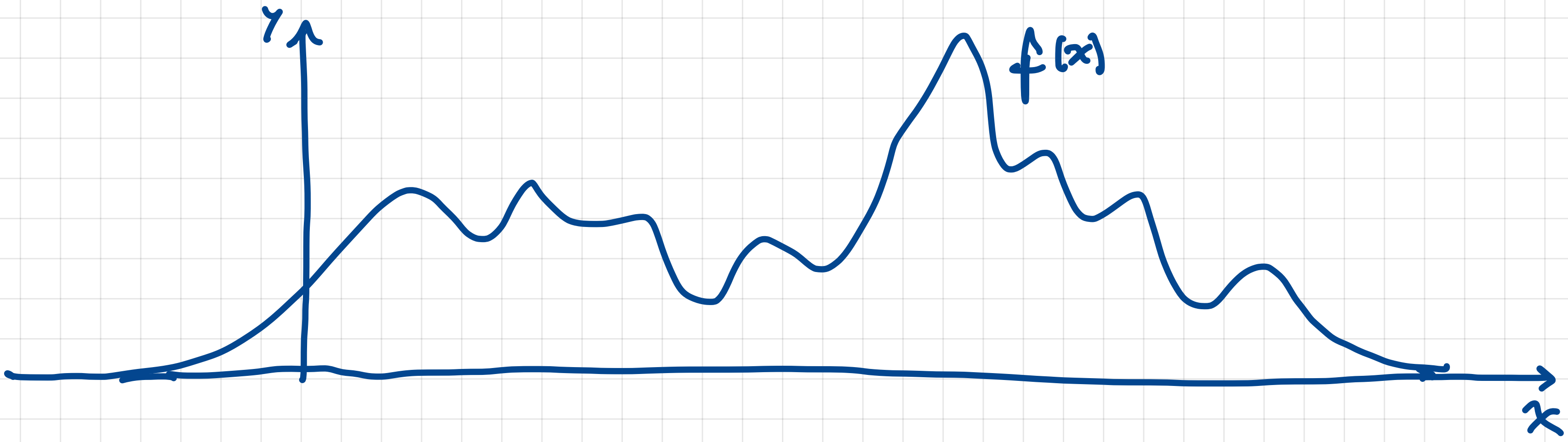
Recall the exponential form of the Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad , \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$$

when extending from $[-\pi, \pi]$ to the whole real line...

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad (1)$$

"Fourier transform"



The Fourier transform analyzes this non-periodic signal giving us its decomposition into Fourier modes ...

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk \qquad (2)$$

sum amplitude mode

"inverse Fourier transform"

Recall the $\dfrac{e^{ikx}}{\sqrt{2\pi}}$ modes are an orthonormal basis:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\, e^{iqx}\, dx = \delta_{qk}$$

Kronecker delta

What happens to this when $[-\pi,\pi] \to (-\infty,\infty)$?

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, e^{iqx}\, dx = \delta(k-q)$$

Dirac delta

We need this to have consistency between (1) & (2):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-iqx}\, dx \overset{(2)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk\right] e^{-iqx}\, dx =$$

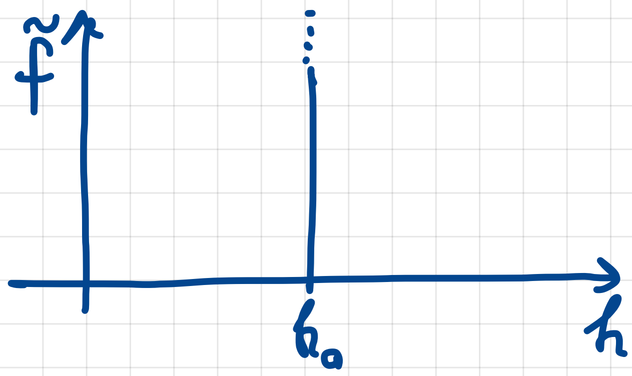
$$= \int_{-\infty}^{\infty} \tilde{f}(k) \underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-q)x}\, dx\right]}_{\delta(k-q)} dk = \tilde{f}(q)$$

Two basic examples

① $f(x) = A_0 \, e^{i k_0 x}$ ← just one mode k_0!

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, e^{-ikx} \, dx$$

$$= A_0 \, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k_0)x} \, dx = A_0 \, \delta(k - k_0)$$



② $f(x) = B_0 \, \delta(x - x_0)$ ← concentrated at one point x_0!

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, e^{-ikx} \, dx$$

$$= B_0 \, \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x - x_0) \, e^{-ikx} \, dx = \frac{B_0}{2\pi} \, e^{-ikx_0}$$