Short Takes 331

Green's functions of the Laplacian: eigenfunction expansion

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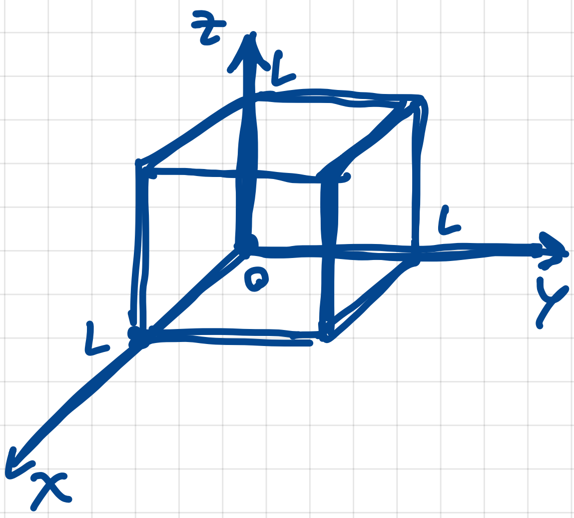
$$\nabla_{\vec r}^2 G(\vec r, \vec r\,') = \delta(\vec r - \vec r\,') \qquad \& \qquad \text{b.c}$$

In the last few videos we used $\nabla^2 \phi_{\vec n} = \lambda_{\vec n} \phi_{\vec n}$ for...

- Cube (Cartesian coords)

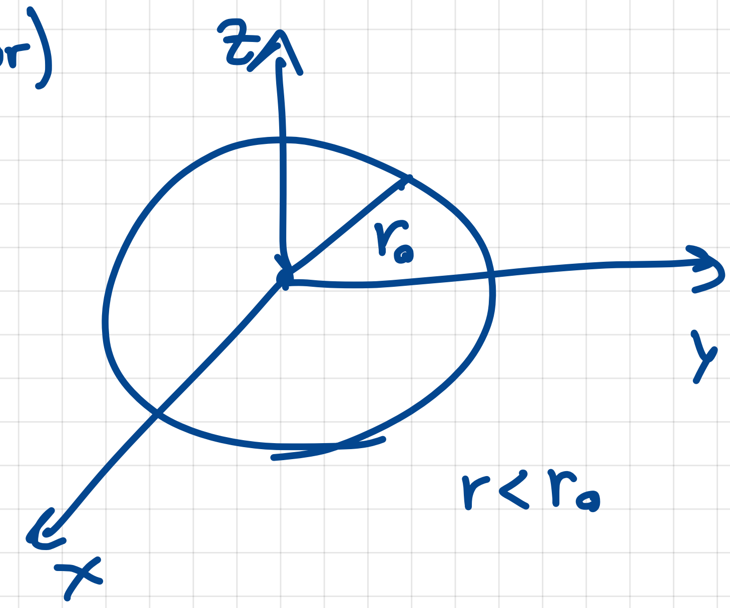


- $f(\vec r, t) = 0$ on faces of the cube.

$$\phi_{\vec n}(\vec r) = \phi_{n_1}(x)\,\phi_{n_2}(y)\,\phi_{n_3}(z), \quad \vec n = (n_1, n_2, n_3), \quad \phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

$$\lambda_{\vec n} = \lambda_{n_1} + \lambda_{n_2} + \lambda_{n_3}, \quad n_i = 1, 2, 3, \ldots, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

- Sphere (interior) (spherical coords)



- $f(\vec r, t)\big|_{|\vec r| = r_0} = 0$

$$\phi_{\vec n}(\vec r) = R_{\ell n}(r)\, Y_{\ell m}(\theta, \varphi), \quad \vec n = (n, \ell, m)$$

$$\lambda_{\vec n} = -\left(\frac{x_{\ell n}}{r_0}\right)^2 \qquad n = 1, \ldots \qquad \ell = 0, 1, \ldots \qquad m = -\ell, \ldots, \ell$$

$R_{\ell n}(r) = j_\ell\left(x_{\ell n}\frac{r}{r_0}\right)$, $\quad x_{\ell n}$: n-th (non-zero) root of $j_\ell(x)$.

"spherical Bessel"

As we've seen in previous videos,

$$G(\vec{r},\vec{r}\,') = \sum_{\vec{n}} \frac{\phi^*_{\vec{n}}(\vec{r}\,')\,\phi_{\vec{n}}(\vec{r})}{\lambda_{\vec{n}}} ,$$

since

$$\nabla^2_{\vec{r}} G(\vec{r},\vec{r}\,') = \sum_{\vec{n}} \frac{\phi^*_{\vec{n}}(\vec{r}\,')\,\nabla^2_{\vec{r}}\phi_{\vec{n}}(\vec{r})}{\lambda_{\vec{n}}}$$

$$= \underline{\sum_{\vec{n}} \phi^*_{\vec{n}}(\vec{r}\,')\,\phi_{\vec{n}}(\vec{r}) = \delta(\vec{r}-\vec{r}\,')} \quad \checkmark$$

"<u>completeness relation</u>"

- <u>cube</u>

$$\underbrace{\left[\sqrt{\frac{2}{L}}\sum_{n_1} \sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n_1\pi x'}{L}\right)\right]}_{\delta(x-x')}\underbrace{\left[\sqrt{\frac{2}{L}}\sum_{n_2} \dots\right]}_{\delta(y-y')}\underbrace{\left[\sqrt{\frac{2}{L}}\sum_{n_3} \dots\right]}_{\delta(z-z')}$$

Note G does <u>not</u> factorize, though!

- <u>sphere</u>

$$\sum_{n,l,m} R_{ln}(r)\,Y_{lm}(\theta,\varphi)\,R^*_{ln}(r')\,Y^*_{lm}(\theta',\varphi') = \delta(\vec{r}-\vec{r}\,')$$

much less obvious!

- <u>What happens if we remove the boundaries?</u>

Then,

$$\phi_{\vec{n}}(\vec{r}) \longrightarrow \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{r}} = \phi_{\vec{k}}(\vec{r})$$

$$\lambda_{\vec{n}} \longrightarrow -\vec{k}^2 = \lambda_{\vec{k}}$$

$\vec{k}$ is now continuous! "(k_x, k_y, k_z)"

and

$$G(\vec{r},\vec{r}') = \int d^3k \, \frac{\phi_{\vec{k}}(\vec{r}) \, \phi^*_{\vec{k}}(\vec{r}')}{\lambda_{\vec{k}}}$$

$$= \frac{1}{(2\pi)^3} \int d^3k \, \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{-\vec{k}^2}$$

$$\nabla^2 G(\vec{r},\vec{r}') = \frac{1}{(2\pi)^3} \int d^3k \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = \delta(\vec{r}-\vec{r}')$$

$\rightarrow$ Fourier transforms!