Short Takes 331

Fourier series: generalizations & applications Part 3

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From part 2...

- How to find $R_n(r)$ explicitly?

$$\nabla^2\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2}\nabla^2_{\theta,\varphi}\phi = \lambda\phi \quad , \quad \lambda = -k^2$$

$$\phi(\vec{r}) = R_n(r)\,Y_{\ell m}(\theta,\varphi) \quad , \quad \nabla^2_{\theta,\varphi}Y_{\ell m} = -\ell(\ell+1)\,Y_{\ell m}$$

Then
$$\left[\begin{aligned} &\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR_n) - \frac{\ell(\ell+1)}{r^2}R_n = -k^2R_n \\ &\qquad\qquad\qquad r\in[0,r_0] \\ &R_n(r=r_0) = 0 \end{aligned}\right. \quad \longrightarrow \quad R_{\ell n}$$

- As $r\to 0$, $\quad \frac{\partial^2}{\partial r^2}(rR) - \ell(\ell+1)\frac{R}{r} = 0$

Propose $R(r) = R_0\,r^m$; then

$$(m+1)\,m\,r^{m-1} - \ell(\ell+1)\,r^{m-1} = 0 \Rightarrow \begin{cases} m = \ell \longleftarrow \text{regular at } r=0 \\ \text{or} \\ m = -\ell-1 \quad \otimes \text{ singular!} \end{cases}$$

- More generally,

$$r\frac{\partial^2}{\partial r^2}(rR) - \ell(\ell+1)R + k^2r^2R = 0$$

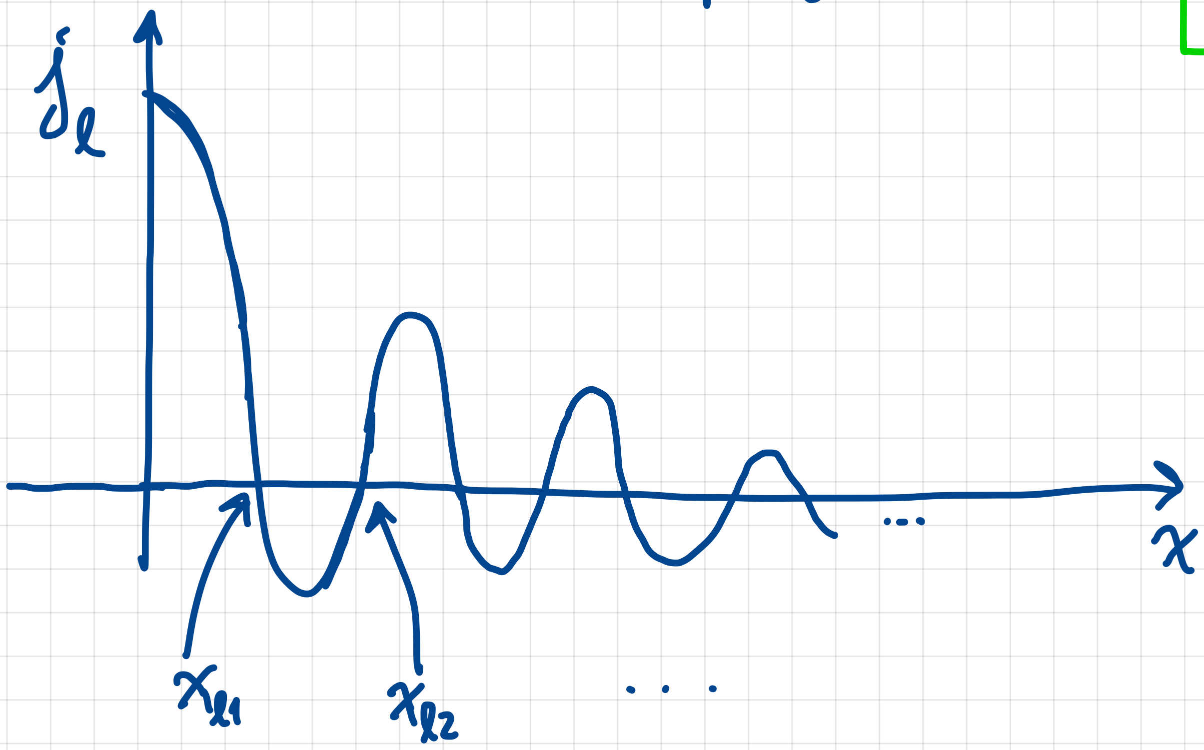
$\rightarrow$ Spherical Bessel functions

$$R_{\ell n}(r) \longrightarrow j_\ell(kr)$$

[also $y_\ell(kr)$
but its singular at $r=0$.

$\rightarrow$ We need $R_{\ell n}(r=r_0) = 0$ $\leftarrow$ boundary condition!

$x_{\ell n}$ = n-th root of $j_\ell(x)$ $\rightarrow$ $\boxed{k_{\ell n} = \frac{x_{\ell n}}{r_0}}$



$$\Rightarrow \boxed{R_{\ell n}(r) = j_\ell(k_{\ell n} r)} = j_\ell\left(x_{\ell n}\frac{r}{r_0}\right)$$

$\ell = 0$: $x_{01} = \pi$, $x_{02} = 2\pi$, ...

$\ell = 1$: $x_{11} = 4.49...$, $x_{12} = 7.72...$, ...

⋮

Note: for $\ell > 0$, avoid $x_{\ell 0} = 0$

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x - x\cos x}{x^2}$$

⋮

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell j_0(x)$$

• Recall our objective ...

$$f(\vec{r},t) = \sum_{n,l,m} C_{\vec{n}}(0)\, e^{D\lambda_{\vec{n}} t}\, R_{ln}(r)\, Y_{lm}(\theta,\varphi)$$

$$\lambda_{\vec{n}} = -k_{ln}^2 = -\left(\frac{x_{ln}}{r_0}\right)^2$$

where we assumed we are given the initial shape

$$f(\vec{r},t=0) = \sum_{n,l,m} C_{\vec{n}}(0)\, R_{ln}(r)\, Y_{lm}(\theta,\varphi)$$

• E.g. a <u>spherically symmetric case</u> as initial condition:

$$C_{nlm}(0) = 0 \quad \forall l > 0 \quad \rightarrow \text{spherical sym.}$$

$$C_{n00}(0) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases} \qquad \text{simplicity}$$

Then, $f(\vec{r},0) = R_{01}(r) = \dfrac{\sin\left(\pi \frac{r}{r_0}\right)}{\pi \frac{r}{r_0}}$

$$f(\vec{r},t) = e^{-D\left(\frac{\pi}{r_0}\right)^2 t} f(\vec{r},0)$$

- Plot it!
- Play with varying n, l
- Add more modes to the initial condition.