Short Takes 331

Fourier series: generalizations & applications Part 2

Fourier series : generalizations & applications. Part 2.

From part 1...

Diffusion in 1D

$\rightarrow$ $$\frac{\partial f(x,t)}{\partial t} = D \frac{\partial^2}{\partial x^2} f(x,t) \quad , \quad f(0,t) = f(L,t) = 0 \quad \forall t$$

0   L

Look for eigenfunctions of $\frac{\partial^2}{\partial x^2}$

$$\frac{\partial^2 \phi_n}{\partial x^2} = \lambda_n \phi_n$$

boundary + initial conditions

$f(x,0) \ \forall x$

with the right boundary conditions...

$$\begin{cases} \phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) , \\ \lambda_n = -\left(\frac{n\pi}{L}\right)^2 , \quad n = 1, 2, 3, \dots \end{cases}$$

- Propose

$$f(x,t) = \sum_{n=1}^{\infty} C_n(t)\, \phi_n(x)$$

$$\sum_{n=1}^{\infty} \frac{\partial C_n(t)}{\partial t} \phi_n(x) = \sum_{n=1}^{\infty} D\, C_n(t) \frac{\partial^2 \phi_n(x)}{\partial x^2}$$

$$= \sum_{n=1}^{\infty} D \lambda_n C_n(t)\, \phi_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{\partial C_n(t)}{\partial t} - D \lambda_n C_n(t) \right) \phi_n(x) = 0$$

$$\Rightarrow \frac{\partial C_n(t)}{\partial t} = D \lambda_n C_n(t)$$

all linearly indep.!

We need initial condition!

$$C_n(t) = C_n(0)\, e^{+D\lambda_n t} \longleftarrow \sum_{n=1}^{\infty} C_n(0)\, \phi_n(x)$$

$$= C_n(0)\, e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

Therefore,

$$f(x,t) = \sum_{n=1}^{\infty} C_n(0)\, e^{-D\left(\frac{n\pi}{L}\right)^2 t} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

I.C.   Diff eq   Diff eq + B.C.

- What happens in higher dimensions? (say 3D)

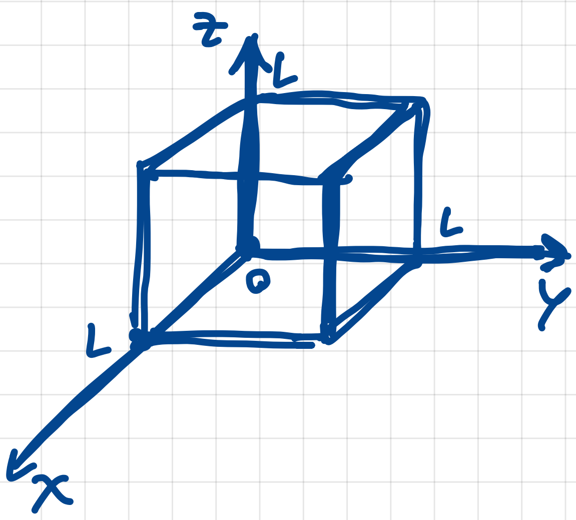
→ If diffusion is isotropic

$$\frac{\partial f}{\partial t}(\vec{r},t) = D\, \nabla^2 f(\vec{r},t) \quad , \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian op.

→ Depending on the boundary conditions (esp. the shape!), we may want to switch coordinates; e.g.

- Cube (Cartesian coords)



- $f(\vec{r},t) = 0$ on faces of the cube.
- $f(\vec{r},0)$ given $\forall \vec{r}$.

$$\begin{cases} \phi_{\vec{n}}(\vec{r}) = \phi_{n_1}(x)\, \phi_{n_2}(y)\, \phi_{n_3}(z) \quad , \quad \vec{n} = (n_1, n_2, n_3) \\ \lambda_{\vec{n}} = \lambda_{n_1} + \lambda_{n_2} + \lambda_{n_3} \qquad n_i = 1, 2, 3, \ldots \end{cases}$$

Repeating our exercise, propose

$$f(\vec{r},t) = \sum_{n_1 n_2 n_3}^{\infty} C_{\vec{n}}(t)\, \phi_{\vec{n}}(\vec{r})$$

Find
$$C_{\vec{n}}(t) = C_{\vec{n}}(0)\, e^{-D\frac{\pi^2}{L^2}(n_1^2+n_2^2+n_3^2)t}$$

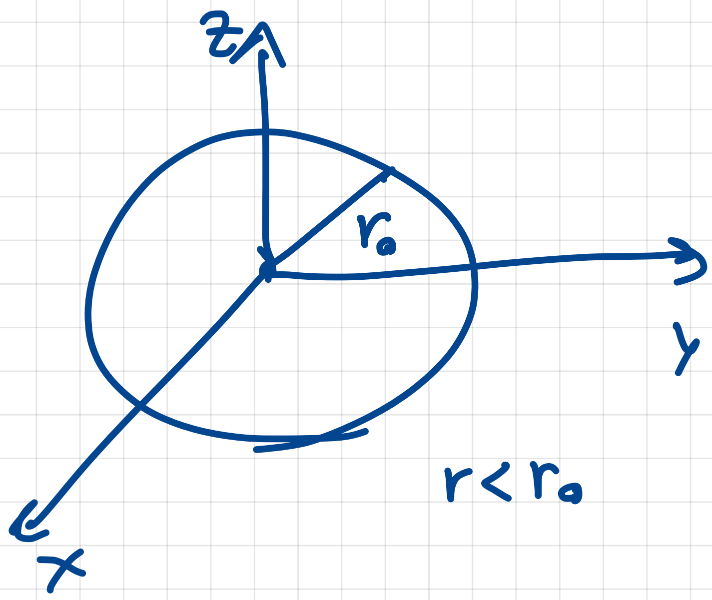
and finally

$$f(\vec{r},t) = \sum_{n_1,n_2,n_3}^{\infty} C_{\vec{n}}(0)\, f_{n_1}(x,t)\, f_{n_2}(y,t)\, f_{n_3}(z,t) \quad ,$$

where $f_n(w,t) = e^{-D\frac{\pi^2}{L^2}n^2 t}\sqrt{\frac{2}{L}}\,\sin\!\left(\frac{n\pi}{L}w\right)$

- sphere

(spherical coords)



$r < r_0$

- $f(\vec{r},t)\big|_{|\vec{r}|=r_0} = 0 \qquad \forall t$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

Here, $\vec{r} = (x,y,z) \longrightarrow (r, \theta, \varphi)$

→ We need eigenfunctions of ∇^2 with the above b.c.

$$\begin{cases} \phi_{\vec{n}}(\vec{r}) = R_n(r)\, Y_{\ell m}(\theta,\varphi) \;, & \vec{n} = (n, \ell, m) \\ & n = 1, \dots \\ \lambda_{\vec{n}} = \dots ? & \ell = 0, 1, \dots \;, \quad m = -\ell, \dots, \ell \end{cases}$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2}\nabla^2_{\theta,\varphi}\phi = \lambda\phi$$

$$\nabla^2_{\theta,\varphi} Y_{lm} = -l(l+1) Y_{lm}$$

$l = 0, 1, \dots$

$m = -l, \dots, l$

$$= \underline{Y_{lm}}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR_n) - \frac{l(l+1)}{r^2}R_n\right] = \lambda R_n \underline{Y_{lm}}$$

Then
$$\begin{cases} \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2}(rR_n) - \dfrac{l(l+1)}{r^2}R_n = \lambda R_n \\ \qquad r \in [0, r_0] \\ R_n(r=r_0) = 0 \quad \& \text{ regular at } r=0. \end{cases}$$

- Then, repeating previous derivation,

$$f(\vec{r},t) = \sum_{n,l,m} C_{\vec{n}}(0)\, e^{+D\lambda_{\vec{n}} t} R_n(r)\, Y_{lm}(\theta,\varphi)$$

$\lambda < 0$, set $\lambda = -k^2$ for some k.