Short Takes 331

Fourier series: generalizations & applications Part 1

Fourier series : generalization & application. Part 1.

- We saw that

$$f(x) = \sum_{k=-\infty}^{\infty} C_k \, e^{i\frac{k\pi x}{L}} \quad , \quad x \in [-L, L]$$

completeness? convergence? $\qquad C_k = \left(\frac{e^{i\frac{k\pi x}{L}}}{2L} , f(x) \right)$

- More generally,

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

"generalized Fourier series"

$F = \{ \phi_0(x), \phi_1(x), \dots \}$

Some convenient orthonormal basis.

$$a_n = (\phi_n(x), f(x)) \longleftarrow (\phi_k , \phi_{k'}) = \delta_{kk'}$$

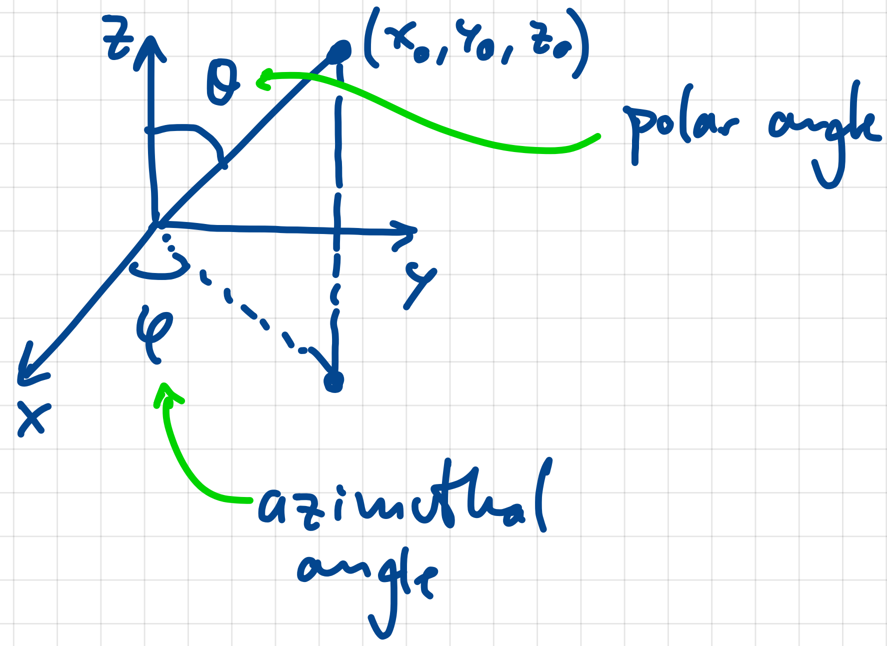
- Examples

① $f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_{lm}(\theta,\varphi)$

"spherical harmonics"

$\theta \in [0, \pi)$

$\varphi \in (0, 2\pi)$



$$\int_0^{\pi}\int_0^{2\pi} Y_{lm}(\theta,\varphi)\, Y_{l'm'}(\theta,\varphi)\, d\Omega = \delta_{ll'}\,\delta_{mm'}$$

!

$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$C_{lm} = \int_0^{\pi}\int_0^{2\pi} f(\theta,\varphi)\, Y_{lm}(\theta,\varphi)\, d\Omega$$

2 $$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

Chebyshev polynomials

$x \in [-1, 1]$

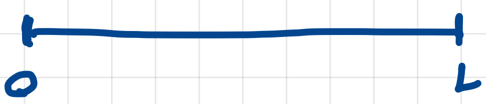
$$\int_{-1}^{1} T_n(x)\, T_m(x)\, (1-x^2)^{-1/2}\, dx = \frac{\pi}{2}\,\delta_{nm}$$

$$\left(= \pi \quad \text{if } n = m = 0\right)$$

$$a_n = \int_{-1}^{1} T_n(x)\, f(x)\, (1-x^2)^{-1/2}\, dx$$

Application : Diffusion in 1D

$$\rightarrow \quad \frac{\partial f(x,t)}{\partial t} = D \frac{\partial^2}{\partial x^2} f(x,t)$$



boundary + initial conditions

$f(x,0)$

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

$$\frac{\partial^2 \phi_n}{\partial x^2} = \lambda_n \phi_n, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

Propose

$$f(x,t) = \sum_{n=1}^{\infty} C_n(t)\,\phi_n(x)$$

$$\sum_{n=1}^{\infty} C_n(0)\,\phi_n(x)$$

$$\Rightarrow \quad \frac{\partial C_n(t)}{\partial t} = D.\lambda_n C_n(t)$$

We need initial condition!

$$C_n(t) = C_n(0)\, e^{+D\lambda_n t} = C_n(0)\, e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

Therefore,

$$f(x,t) = \sum_{n=1}^{\infty} C_n(0)\, e^{-D\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

I.C. Diff eq Diff eq + B.C.