Short Takes 331

Calculating roots: the bisection algorithm

Root finding: part 1.

Root finding is a misnomer → "root-calculating"

$P(x) = 0 \quad \rightarrow \quad x = ?$

- $P(x)$ quadratic polynomial → ok
  cubic
  quartic

$$P(x) = \sum_{k=0}^{N} a_k x^k \qquad \deg P \leq N$$

Two theorems

- Abel-Ruffini: concerning the impossibility of solving for the roots "in terms of radicals" for degree 5 and higher.

- Fundamental theorem of algebra: All the roots are in $\mathbb{C}$, for all polynomials.

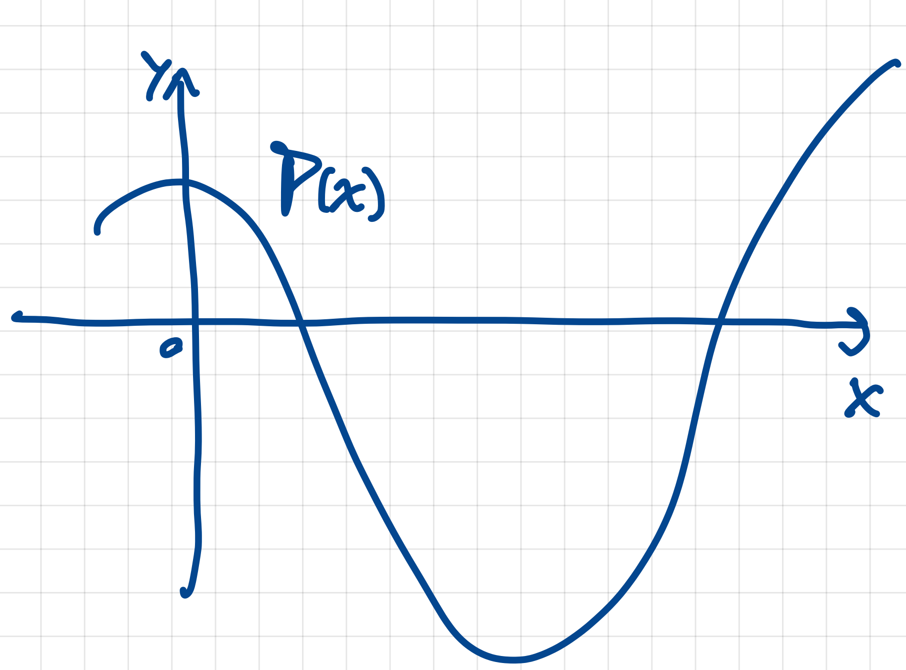
- $P(x)$ could be more complicated

  e.g. $P(x) = e^x - 7\sin(x)$

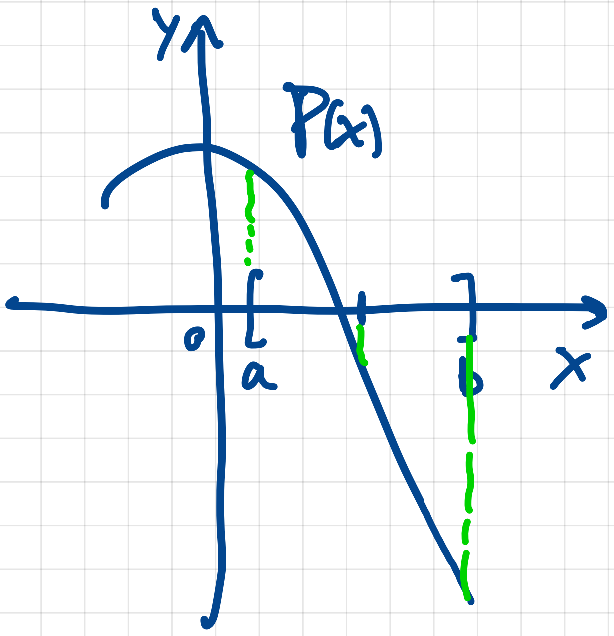
→ We need numerical methods!

Bisection (a modern approach)

Step 0. Plot the function!



Step 1. Bracket the root you want.



→ root is between a and b.

→ check P(a), P(b) have different signs.

step 2. Estimate root as

$$x_r = \frac{a+b}{2}$$ ← calculate this.

step 3. Calculate

$P(a)$, $P(x_r)$, $P(b)$

- If $P(a)$ and $P(x_r)$ have same sign → shift a to x_r.
- If $P(b)$ and $P(x_r)$ have same sign → shift b to x_r.

- Return to step 2 with the updated values of a or b.

- When to stop?

After N steps, the size of the bracket will have gone down by 2^N.

$$\text{size} = b-a \xrightarrow{N \text{ steps}} \frac{b-a}{2^N}$$

If we want a bound of size Δ (ie error $\frac{\Delta}{2}$) then

$$\frac{b-a}{2^N} = \Delta \Rightarrow \frac{b-a}{\Delta} = 2^N$$

$$\Rightarrow N = \log_2\left(\frac{b-a}{\Delta}\right)$$

- Caveats: multiple roots

- It looks like we can't even start the algorithm because P(x) has the same sign on both sides of the root

What to do?