Short Takes 331

The geometric sum & series

Geometric sum and series.

Geometric sum: $S_N = \sum_{k=0}^{N} x^k$

Geometric series: $\lim_{N\to\infty} S_N = \sum_{k=0}^{\infty} x^k$

$S_N = ?$

$$S_N = 1 + x + x^2 + \dots + x^N$$

$$x S_N = x + x^2 + \dots + x^{N+1}$$

(1) $$1 + x S_N = 1 + x + x^2 + \dots + x^{N+1} = S_{N+1}$$

but ...

(2) $$S_{N+1} = 1 + x + \dots + x^N + x^{N+1} = S_N + x^{N+1}$$

$$\Rightarrow \quad 1 + x S_N = S_N + x^{N+1}$$

$$\Rightarrow \quad 1 - x^{N+1} = S_N (1 - x)$$

$$\Rightarrow \quad \boxed{S_N = \frac{1 - x^{N+1}}{1 - x}} \qquad \text{any } x \neq 1$$

If $|x| < 1 \quad \Rightarrow \quad S_N \xrightarrow{N\to\infty} \frac{1}{1-x}$

because $x^{N+1} \to 0$