Short Takes 331

The binomial theorem

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\frac{n!}{k!(n-k)!}$$ "binomial coefficient"

- **By induction!**
  - Prove $n=1$

$$(a+b)^1 = a+b = \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k}$$

$$= \binom{1}{0} b + \binom{1}{1} a$$

$$\qquad 1 \qquad\qquad 1$$

✓

  - Prove case n implies case $n+1$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

let's see ...

$$(a+b)^{n+1} = (a+b)^n (a+b)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n-k+1} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$= \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] a^k b^{n+1-k} + b^{n+1} + a^{n+1}$$

$$\frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} = n!\left(\frac{k + n-k+1}{k!\,(n-k+1)!}\right)$$

$(k-1)!$: $k!=k.(k-1)!$

$(n-k+1)! = (n-k+1).(n-k)!$

$$= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}$$

So far ...

$$(a+b)^{n+1} = \sum_{k=1}^{n}\binom{n+1}{k}a^k\,b^{n+1-k} + \underset{k=0}{b^{n+1}} + \underset{k=n+1}{a^{n+1}}$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k\,b^{n+1-k} \quad ✓$$

- A few useful remarks ...

- $\underbrace{(a+b)(a+b)\dots(a+b)}_{n \text{ times}}$

$a^n \longrightarrow 1$ time

$a^{n-1}b \longrightarrow n$ times

$\vdots$

$a\,b^{n-1} \longrightarrow n$ times

$b^n \longrightarrow 1$ time

$\longrightarrow$ check out Tartaglia's triangle!

- $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$

- $(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n\sum_{k=0}^{n}\binom{n}{k}b^k a^{-k}$