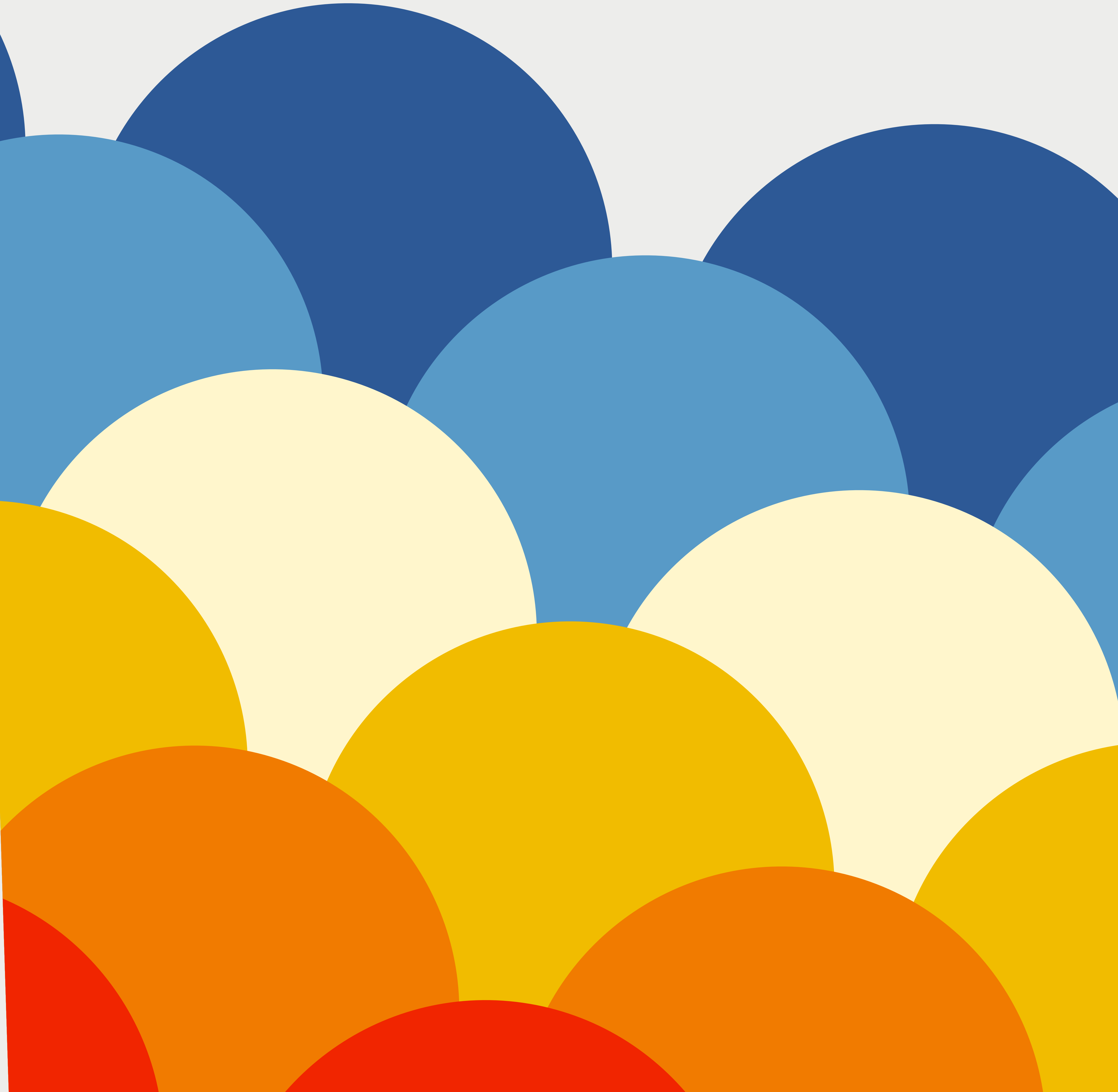
Short Takes 331

Fourier series: [sin(x)]^n



Fourier series : $\sin^n(x)$

Exponential form:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \, e^{ikx} \quad , \quad x \in [-\pi, \pi]$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-ikx} \, dx$$

without integrals?

$$(\sin(x))^n = \left[\frac{e^{ix} - e^{-ix}}{2i} \right]^n$$

binomial theorem: $(a+b)^n = \sum_{q=0}^{n} \binom{n}{q} a^q \, b^{n-q}$

$$\binom{n}{q} = \frac{n!}{q!\,(n-q)!}$$

$$= \left(\frac{1}{2i}\right)^n \sum_{q=0}^{n} \binom{n}{q} (-1)^q \, e^{-iqx} \quad e^{i(n-q)x}$$

$$= \frac{1}{2^n} \cdot (-1)^n \, i^n \sum_{q=0}^{n} \binom{n}{q} (-1)^q e^{i(n-2q)x}$$

$k = n - 2q \quad \rightarrow \quad q = 0 \Rightarrow k = n$

$q = 1 \Rightarrow k = n - 2$

$q = 2 \Rightarrow k = n - 4$

$\vdots$

$q = n \Rightarrow k = n - 2n = -n$

$$= \sum_{k=-n}^{n} c_k e^{ikx} \quad , \quad c_n = \frac{1}{2^n}(-1)^n i^n \binom{n}{0}$$

$$c_{n-1} = 0$$

$$c_{n-2} = \frac{1}{2^n}(-1)^{n+1} i^n \binom{n}{1}$$

$$c_{n-3} = 0$$

$$\vdots$$

$$c_{-n+1} = 0$$

$$c_{-n} = \frac{1}{2^n} i^n \binom{n}{n}$$

- The highest freq modes are $\pm n$.
- The series is actually a finite sum, i.e. it "terminates".
- Half of the coefficients between $k=-n$ and $k=n$ vanish.
- We can check $c_{-k} = c_k^*$ since the full result must be real.

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \Rightarrow f^*(x) = \sum_{k=-\infty}^{\infty} c_k^* e^{-ikx}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k}^* e^{ikx}$$

But if $f(x) \in \mathbb{R}$ then $f^*(x) = f(x)$,

and so $c_{-k} = c_k^*$.