Short Takes 331

Fourier series: exponential form

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- Recall

$$\boxed{f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right]}$$

and also

$$\begin{cases} e^{iy} = \cos y + i\sin y \\ e^{-iy} = \cos y - i \sin y \end{cases} \Rightarrow \begin{cases} \cos y = \dfrac{e^{iy} + e^{-iy}}{2} \\ \sin y = \dfrac{e^{iy} - e^{-iy}}{2i} \end{cases}$$

This shows there exist coefficients C_n (related to A_n, B_n) such that

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{n\pi x}{L}\right)}}$$

- How are they related?

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{n\pi x}{L}\right)} = C_0 + \sum_{n=1}^{\infty} C_n e^{i\left(\frac{n\pi x}{L}\right)} + \sum_{n=1}^{\infty} C_{-n} e^{-i\left(\frac{n\pi x}{L}\right)}$$

$$= C_0 + \sum_{n=1}^{\infty}\left(C_n e^{i\left(\frac{n\pi x}{L}\right)} + C_{-n} e^{-i\left(\frac{n\pi x}{L}\right)}\right)$$

$$= \underbrace{C_0}_{\frac{A_0}{2}} + \sum_{n=1}^{\infty}\left[\underbrace{(C_n + C_{-n})}_{A_n}\cos\left(\frac{n\pi x}{L}\right) + \underbrace{i(C_n - C_{-n})}_{B_n}\sin\left(\frac{n\pi x}{L}\right)\right]$$

Inverting...
$$\begin{cases} C_k = \dfrac{A_k - iB_k}{2} & k \geq 1 \\ C_k = \dfrac{A_{-k} + iB_{-k}}{2} & k \leq -1 \end{cases}$$

Note: $B_k = 0 \Rightarrow C_k = C_{-k}$

$A_k = 0 \Rightarrow C_k = -C_{-k}$

- Also,
$$\boxed{C_k = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\left(\frac{k\pi x}{L}\right)}\, dx}$$

Check for consistency...

$$C_q \overset{?}{=} \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\left(\frac{q\pi x}{L}\right)}\, dx = \sum_{k=-\infty}^{\infty} C_k \underbrace{\frac{1}{2L}\int_{-L}^{L} e^{i\frac{\pi x}{L}(k-q)}\, dx}_{\delta_{kq}}$$

- $k = q$: $\frac{1}{2L}\cdot\int_{-L}^{L} dx = 1$ ✓
- $k \neq q$: $\frac{1}{2L}\cdot\frac{L}{i\pi}\cdot\frac{1}{k-q}\; e^{i\frac{\pi x}{L}(k-q)}\Big|_{-L}^{L} = 0$ ✓

$$e^{i\pi m} = e^{-i\pi m} = (-1)^m$$