Short Takes 331

Eigenvectors & Eigenvalues Part 2

Eigenvalues & Eigenvectors. Part 2.

- From part 1 ...

  - Defining eq: $A\vec{v} = \lambda \vec{v}$, $\vec{v} \neq 0$, A square matrix. ($\mathbb{R}$ or $\mathbb{C}$)

  - Secular eq: $\det(A - \lambda \mathbb{1}) = 0 \longrightarrow \lambda$
    (or characteristic eq.)
    characteristic polynomial

  - Normal matrices: $A^\dagger A = A A^\dagger \iff [A^\dagger, A] = 0$
    "commutator"

  - Theorem: A is normal iff
    $\exists\, U$ unitary (i.e. $U^\dagger U = \mathbb{1}$)
    such that

    $$U^\dagger A U = D$$

    diagonal matrix of eigenvalues (spectrum)

    contains eigenvectors as columns $U = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_N \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{pmatrix}$

- Matrix diagonalization

  - Find eigenvectors $\vec{v}_i$ of A
  - Construct U
  - calculate

    $$U^\dagger A U = D$$

Note: $\underbrace{U U^\dagger}_{\mathbb{1}} A \underbrace{U U^\dagger}_{\mathbb{1}} = U D U^\dagger \Rightarrow A = U D U^\dagger$

"spectral decomposition"
(or spectral representation)

- What if A is not normal?

If A has a complete set of eigenvectors (i.e. they form a basis), then we can form a matrix

$$V = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$ which has an inverse V^{-1} $(\neq V^{\dagger})$

and so

$$AV = VD \quad \text{(by def. of eigenvectors)}$$

$$\Rightarrow \underbrace{V^{-1}AV}_{} = D \qquad \text{we also have } A = VDV^{-1}$$

↳ a type similarity transformation.

- Matrices that are "similar" to a diagonal matrix are said to be "diagonalizable".
- Normal matrices are a particular case of this; there are diagonalizable matrices that are not normal.
- Non-diagonalizable square matrices are called "defective".

- Application

$$\exp(A) = \exp(VDV^{-1})$$

$$= \mathbb{1} + VDV^{-1} + \frac{1}{2!}(VDV^{-1})(VDV^{-1}) + \dots$$

$$= VV^{-1} + VDV^{-1} + \frac{1}{2!}VD^2V^{-1} + \dots$$

$$= V\left(\mathbb{1} + D + \frac{D^2}{2!} + \dots\right)V^{-1} = V e^D V^{-1}$$

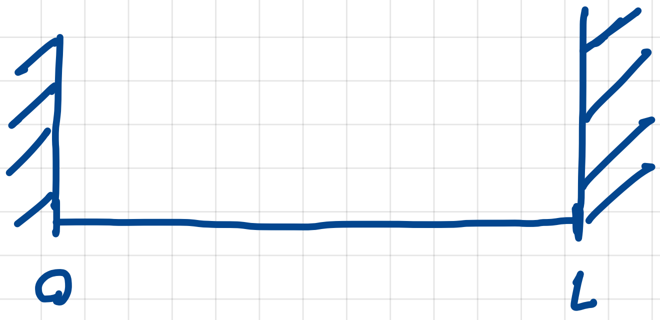
True in general using any Taylor expansion:

$$f(A) = V f(D) V^{-1}$$

- **Another application**

Schrödinger eq. $\quad \underbrace{-\nabla^2}_{\text{operator}} \psi(x) = \underbrace{2mE}_{\text{eigenvalue}} \psi(x)$ $\longrightarrow$ eigenvector

In 1D : $\quad -\dfrac{d^2\psi}{dx^2} = 2mE\psi$



Boundary conditions
$\phi(0) = \phi(L) = 0$

Solution:

$$\psi(x) = C_1 \sin(kx) + C_2 \cos(kx)$$

$\longrightarrow$ 0

$\longrightarrow \quad -\psi''(x) = +k^2 \psi(x)$

$\Rightarrow \quad k^2 = +2mE$

Moreover, $\quad \phi(L) = 0 \Rightarrow kL = n\pi \quad , \; n = 1, 2, \ldots$

(not $n=0$?)

Therefore, $\quad k^2 = \left(\dfrac{n\pi}{L}\right)^2 = 2mE \quad \Rightarrow \quad E = \left(\dfrac{n\pi}{L}\right)^2 \cdot \dfrac{1}{2m}$

$$\begin{cases} \psi_n = C \sin(k_n x) \;, & x \in [0, L] \\ E_n = \dfrac{k_n^2}{2m} & \\ k_n = \dfrac{n\pi}{L} \;, & n = 1, 2, 3, \ldots \end{cases}$$