Short Takes 331

Eigenvectors
&
Eigenvalues

Eigenvalues & Eigenvectors. Part 1.

Definition: Given matrix A ...

$$\underbrace{A\vec{v}}_{\text{eigenvector } (\neq \vec{0})} = \underbrace{\lambda}_{\text{eigenvalue } \to \text{ a scalar}}\vec{v}$$

"When applying A to $\vec{v}$, we get a constant times $\vec{v}$"

→ Note: Given A, this only happens for specific values of λ & specific $\vec{v}$, which vary with λ.

- How to calculate them?

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$0 = A\vec{v} - \lambda\vec{v} = (A - \lambda\mathbb{1})\vec{v}$$

$$\underbrace{\boxed{\det(A - \lambda\mathbb{1}) = 0}}_{\text{"secular equ."}}$$

"characteristic polynomial"

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - cb = 0$$

$\lambda = ?$

$$\lambda^2 - (a+d)\lambda + ad - cb = 0$$

→ quadratic eq. for λ

λ_1 & λ_2 eigenvalues

Eigenvectors

$\vec{v}_1 \quad \lambda_1 \quad \longrightarrow \quad \begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = 0$

$\vec{v}_2 \quad \lambda_2$

$$(a-\lambda_1)v_{1x} + b\,v_{1y} = 0$$

$$\Rightarrow v_{1y} = -\frac{(a-\lambda_1)}{b}v_{1x} = \left(\frac{\lambda_1 - a}{b}\right)v_{1x}$$

$$\vec{v}_1 = \begin{pmatrix} v_{1x} \\ \frac{\lambda_1 - a}{b}\,v_{1x} \end{pmatrix} = v_{1x}\begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix}$$

$$|\vec{V}_1|^2 = 1 \qquad \text{"Normalization"}$$

$$V_{1x}^2\left(1 + \left(\frac{\lambda_1 - a}{b}\right)^2\right) = 1 \longrightarrow V_{1x} = \sqrt{\frac{1}{1+\left(\frac{\lambda_1 - a}{b}\right)^2}}$$

Similarly, $\vec{V}_2 = V_{2x}\begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix}$, $V_{2x} = \sqrt{\frac{1}{1+\left(\frac{\lambda_2 - a}{b}\right)^2}}$

Example

$$H = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix} \qquad H^{+} = (H^{T})^{*} = H$$

$$\det(H - \lambda \mathbb{1}) = \begin{vmatrix} 1-\lambda & i \\ -i & -\lambda \end{vmatrix} = \lambda(\lambda - 1) - 1 = 0 \Rightarrow$$

$$\lambda^2 - \lambda - 1 = 0 \Rightarrow \begin{cases} \lambda_1 = \frac{1+\sqrt{5}}{2} \\ \lambda_2 = \frac{1-\sqrt{5}}{2} \end{cases}$$

both $\in \mathbb{R}$!

$$\vec{V}_1 = V_{1x}\begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix} , \quad \alpha_1 = \frac{\lambda_1 - 1}{i}$$

$$\vec{V}_2 = V_{2x}\begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix} , \quad \alpha_2 = \frac{\lambda_2 - 1}{i}$$

Note: $\vec{V}_1 \cdot \vec{V}_2^{*} = V_{1x} V_{2x}^{*} (1 + \alpha_1 \alpha_2^{*})$

$$\alpha_1 \alpha_2^{*} = (+1)\cdot(\lambda_1 - 1)(\lambda_2 - 1)$$

$$(\lambda_1 - 1)(\lambda_2 - 1) = \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)\left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right) = \dots = -1$$

$$\Rightarrow \vec{V}_1 \cdot \vec{V}_2^{*} = 0 .$$

→ General rule: eigenvectors of Hermitian matrices are orthogonal if they correspond to different eigenvalues.

Another example

$$A = \begin{pmatrix} 1 & i \\ 1 & 0 \end{pmatrix} \qquad \det(A - \lambda \mathbb{1}) = \begin{vmatrix} 1-\lambda & i \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda-1) - i$$

→ must be λ complex!

Are the eigenvectors orthogonal to each other? Find out!

Theorem: If A is a normal matrix, meaning $A^\dagger A = A A^\dagger$, then its eigenvectors form a basis.

More precisely: A is normal if and only if it is unitarily equivalent to a diagonal matrix.

$$U^\dagger A U = D$$

The columns of U are orthonormal vectors that form a basis and are the eigenvectors of A.

What about A above?

$$A^\dagger A = \begin{pmatrix} 1 & 1 \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix}$$

$$A A^\dagger = \begin{pmatrix} 1 & i \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$A^\dagger A \neq A A^\dagger \Rightarrow A$ is not normal.

Its eigenvectors may still form a basis, but it won't be orthonormal.