Short Takes 331

Determinants Part 2: Two applications

- Recall the recursive definition (Laplace expansion)

$$\det A = \sum_{j=1}^{N} a_{1j} (-1)^{1+j} \det A_{1j}$$

This motivates us to define the so-called cofactor matrix C

$$[C]_{ij} = (-1)^{i+j} \det A_{ij} .$$

Then,

$$[A.C^T]_{ij} = \sum_{k=1}^{N} a_{ik} C_{jk} = \sum_{k=1}^{N} a_{ik} (-1)^{j+k} \det A_{jk}$$

$$= \begin{cases} \det A & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The same holds for $C^T A$.

Thus, $AC^T = C^T A = \det A . \mathbb{1} \Rightarrow A \dfrac{C^T}{\det A} = \dfrac{C^T}{\det A} A = \mathbb{1}$,

i.e. $A^{-1} = \dfrac{C^T}{\det A}$.

"the inverse is equal to the cofactor^T divided by the determinant".

- You can regard this as the application of Cramer's rule to the problem

$$A.M = \mathbb{1} \Leftrightarrow A\vec{M}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\&\; A\vec{M}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\& \dots$$

$$M = \begin{pmatrix} \vec{M}_1 & \vec{M}_2 & \dots & \vec{M}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \qquad "\vec{E}_1" \qquad "\vec{E}_2"$$

$$[\vec{M}_1]_i = \frac{1}{\det A} (-1)^{1+i} \det A_{1i} , \quad [\vec{M}_2]_i = \frac{1}{\det A} (-1)^{2+i} \det A_{2i} , \dots$$

$$M_{11} = [\vec{M}_1]_1 = \begin{vmatrix} 1 & a_{12} & a_{13} & \cdots \\ 0 & a_{22} & \cdots & \\ 0 & & & \\ \vdots & \vdots & & \end{vmatrix} \cdot \frac{1}{\det A} \qquad \ldots$$

$$M_{21} = [\vec{M}_1]_2 = \begin{vmatrix} a_{11} & 1 & a_{13} & \cdots \\ a_{21} & 0 & a_{23} & \cdots \\ & \vdots & & \end{vmatrix} \frac{1}{\det A}$$

$$\vdots$$

- Given $N \times N$ problem

  $B\vec{x} = 0$, we always have at least $\vec{x} = 0$ as a solution.

  - If $\det B \neq 0$, then $\vec{x} = 0$ is the only solution.
  - If $\det B = 0$, then there is an infinite number of solutions because the columns of B are linearly dependent.

    (not enough constraints to force $\vec{x} = 0$)

    non-vanishing!

* This is useful when calculating the so-called eigenvectors & eigenvalues of a matrix (future video on this!) ...

Eigenvectors of M: nonzero vectors such that $\exists\, \lambda$ (scalar) for which

$$M\vec{v} = \lambda\vec{v}$$

eigenvectors   eigenvalues

→ How do we solve this problem? (find λ & $\vec{v}$)

$(M - \lambda \mathbb{1})\vec{v} = 0 \quad \Rightarrow \quad \vec{v} = 0$??? Not the solution we want!

To have $\vec{v} \neq 0$, it must be that

$$\det(M - \lambda \mathbb{1}) = 0 \longrightarrow \text{"Secular equation"}$$

It fully determines the allowed values of λ.

For example,

$$M = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \Rightarrow (M - \lambda \mathbb{1}) = \begin{pmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{pmatrix}$$

$$\det(M - \lambda \mathbb{1}) = \lambda(\lambda - 1) - 2 \overset{!}{=} 0$$

$\hookrightarrow$ quadratic eq. for λ.