# Short Takes 331

# Determinants. Part 1

Recall Cramer's rule for 2x2 systems...

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \quad x_1 = \frac{\begin{vmatrix} y_1 & b \\ y_2 & d \end{vmatrix}}{\det A} = \frac{y_1 d - y_2 b}{\det A}$$

$$\det A = ad - bc \qquad\qquad x_2 = \frac{\begin{vmatrix} a & y_1 \\ c & y_2 \end{vmatrix}}{\det A} = \frac{a y_2 - c y_1}{\det A}$$

This rule holds for the general $N \times N$ case.

But how are general determinants defined?

- Recursive definition (Laplace expansion)

$$\det A = \sum_{j=1}^{N} a_{1j} (-1)^{1+j} \det A_{1j}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & & & \\ \vdots & & & \\ & & & a_{NN} \end{pmatrix} \quad N \times N \text{ matrix}$$

$A_{ij}$ is obtained from $A$ by removing row $i$ and column $j$.

- Example $3 \times 3$

$$\underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}_{M} = a_{11} \overbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}^{M_{11}} - a_{12} \overbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}^{M_{12}} + a_{13} \overbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}^{M_{13}}$$

$$= \dots$$

- Another definition

$$\det A = \sum_{\substack{i_j=1 \\ j=1,\dots,N}}^{N} \varepsilon_{i_1 i_2 \dots i_N} \, a_{1 i_1} a_{2 i_2} \dots a_{N i_N}$$

Levi-Civita tensor $= \begin{cases} 1 & \text{if } (i_1 i_2 \dots i_N) \text{ is an even permutation of } (1 2 3 \dots N) \\ -1 & \text{if odd permutation} \\ 0 & \text{otherwise} \end{cases}$

E.g. $\varepsilon_{ij} = \begin{cases} 1 & i=1 \quad j=2 \\ -1 & i=2 \quad j=1 \\ 0 & i=j \end{cases}$

$\varepsilon_{ijk} = \dots ?$

- Useful properties

$$\det \mathbb{1} = \varepsilon_{123\dots N} \, \mathbb{1}_{11} \mathbb{1}_{22} \dots \mathbb{1}_{NN} = 1$$

$$\det D = d_{11} d_{22} \dots d_{NN}$$

diagonal matrix

$$\det(c \cdot M) = c^N \cdot \det M$$

$$\det(AB) = \det A \, \det B \qquad \text{but} \qquad \det(A+B) \neq \det A + \det B$$

$$\det A^{-1} = (\det A)^{-1}$$

$$\frac{\partial}{\partial \lambda} \det A(\lambda) = \det A(\lambda) \cdot \mathrm{Tr}\left(A^{-1} \cdot \frac{\partial A}{\partial \lambda}\right)$$

$$\det(\exp A) = \exp(\mathrm{tr} A)$$

If two rows or columns are proportional to each other $\rightarrow \det A = 0$.
If the rows or columns form a linearly-dependent set $\nearrow$

$\det U = d_{11} d_{22} \cdots d_{NN}$

The determinant of an upper (or lower) triangular matrix is the product of its diagonal elements.

This is the way to calculate determinants (via Gaussian elimination). If we use the definition, it requires $N!$ terms.

- Application

Multidimensional Gaussian integrals

$$I_d = \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_d \, \exp\left(-x^T M x\right)$$

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$ $\qquad x^T = (x_1 \, x_2 \cdots x_d)$ $\qquad M :$ $d \times d$ matrix

This integral does not always exist; it depends on $M$, as it may be possible to find some direction in which the exponent runs away to $-\infty$ or is just constant $\rightarrow$ integral diverges!

Eg. $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is ok but $M = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ is not.

When the integral does exist, the result turns out to be

$$I_d = \left(\frac{\pi^d}{\det M}\right)^{1/2}$$