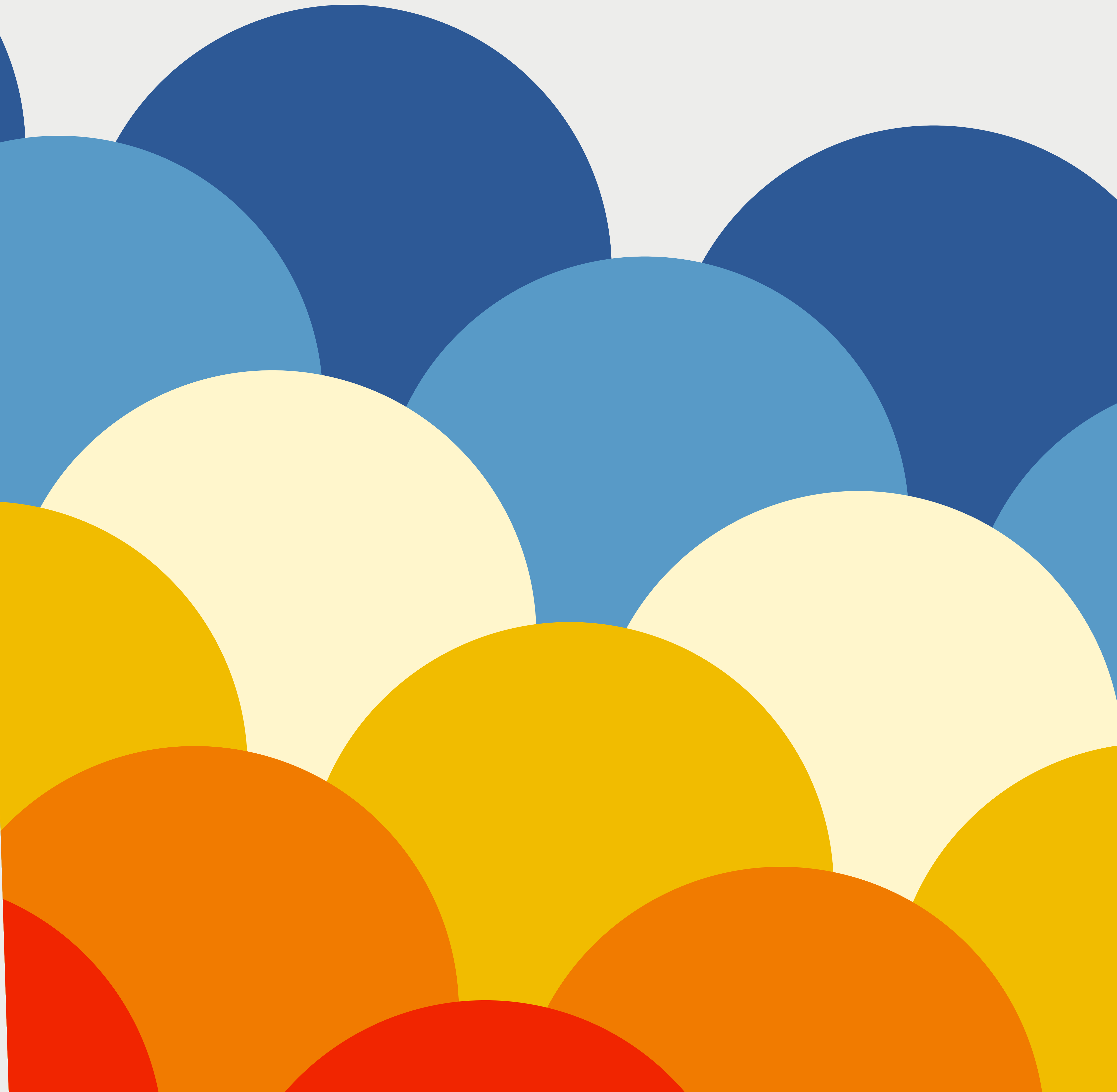
Short Takes 331



Linear operators : Part 3

Theorem : If $\ker A = \{\vec{0}\}$ then A is "one-to-one" (injective) and therefore the inverse operator exists.

→

- Note $A\vec{0} = \vec{0}$ always; therefore, if A is injective then it must be $\ker A = \{\vec{0}\}$ (converse property).

- Assume $\ker A = \{\vec{0}\}$ and A not injective.
Then, $\exists\ \vec{v}_1, \vec{v}_2$ such that

$$\begin{cases} A\vec{v}_1 = \vec{w} \\ A\vec{v}_2 = \vec{w} \end{cases} \underset{\text{linearity}}{\Rightarrow} A(\vec{v}_1 - \vec{v}_2) = \vec{0} \Rightarrow \vec{v}_1 - \vec{v}_2 \in \ker A$$

$$\Rightarrow \vec{v}_1 - \vec{v}_2 = \vec{0}\ , \text{ i.e. } \vec{v}_1 = \vec{v}_2$$

✓

Theorem : The inverse of a linear operator is also a linear op.

→ . By def $A\vec{x} = \vec{y} \Rightarrow A^{-1}\vec{y} = \vec{x}$

$$\vec{x} \underset{A^{-1}}{\overset{A}{\rightleftarrows}} \vec{y}$$

Then $A(c\vec{x}) = c A\vec{x} = c\vec{y}$

$$\Rightarrow A^{-1}(c\vec{y}) = c\vec{x} = c A^{-1}\vec{y}$$

Moreover, $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \vec{w}_1 + \vec{w}_2$

and therefore $A^{-1}(\vec{w}_1 + \vec{w}_2) = \vec{v}_1 + \vec{v}_2 = A^{-1}\vec{w}_1 + A^{-1}\vec{w}_2$

✓

Theorem: The columns of an invertible matrix form a linearly independent set.

$$\longrightarrow \quad A = \begin{pmatrix} \vec{A}_1 & \vec{A}_2 & \vec{A}_3 & \dots & \vec{A}_N \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

$$A\vec{x} = 0 \quad \text{for some } \vec{x} \quad (\text{e.g. } \vec{x} = \vec{0})$$

- If A is invertible, then $\vec{x} = 0$ is the only solution, But ...

$$A\vec{x} = \underbrace{x_1 \vec{A}_1 + x_2 \vec{A}_2 + \dots + x_N \vec{A}_N}_{\text{linear combo of cols of } A} = \vec{0}$$

If the only possible solution is $x_1 = x_2 = \dots = x_N = 0$, then $\{\vec{A}_j \quad j = 1, \dots, N\}$ form an LI set.

- The converse is true as well: if the columns are LI, then $\ker A = \{\vec{0}\}$ and A is invertible. ✓

Example:

$$M = \begin{array}{c} \begin{array}{ccc} \vec{M}_1 & \vec{M}_2 & \vec{M}_3 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix} \end{array}$$

We see $\vec{M}_3 = \vec{M}_1 + \vec{M}_2$, so these three vectors are not LI.

Moreover,

$$M\vec{x} = x_1 \vec{M}_1 + x_2 \vec{M}_2 + x_3 \vec{M}_3 = \vec{0} \quad \text{if} \quad x_1 = x_2 = -x_3$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

indeed, then

$$\text{e.g.} \begin{cases} x_1 = x_2 = 1 \\ x_3 = -1 \end{cases}$$

$$M\vec{x} = -x_3 \underbrace{(\vec{M}_1 + \vec{M}_2 - \vec{M}_3)}_{\vec{0}} = \vec{0}$$