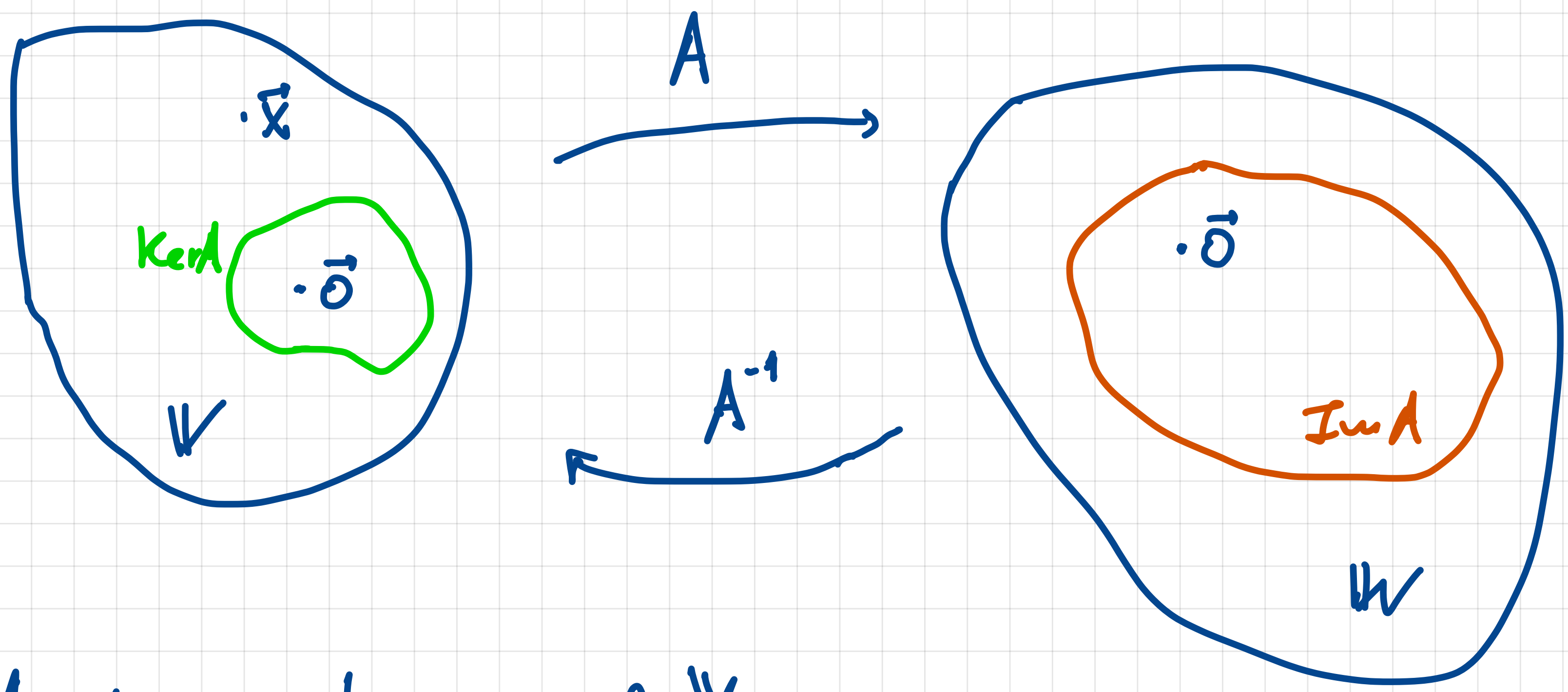
Short Takes 331

Linear operators : Part 2

Definition: A linear operator is a mapping A between to vector spaces $\mathbb{V}$, $\mathbb{W}$ that satisfies two properties:

① $A(\vec{v}_1+\vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$, $\vec{v}_1, \vec{v}_2 \in \mathbb{V}$

② $A(c\vec{v}) = cA\vec{v}$, $\vec{v} \in \mathbb{V}$



Ker A is a subspace of $\mathbb{V}$ → itself a vector space, fully contained in $\mathbb{V}$. It may be just $\{\vec{0}\}$ or all of $\mathbb{V}$.

$\{$All $\vec{x} \in \mathbb{V}$ s.t. $A\vec{x} = \vec{0}\}$

Im A is a subspace of $\mathbb{W}$

$\{$All $\vec{w} \in \mathbb{W}$ s.t. $\exists\, \vec{x} \in \mathbb{V}$ for which $A\vec{x} = \vec{w}\}$

Examples

- $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\mathbb{V} = \mathbb{R}^3$, $\mathbb{W} = \mathbb{R}^3$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Ker}\,A = \mathbb{R}^3, \quad \mathrm{Im}\,A = \{\vec{0}\}$$

- $A: \mathbb{C}^2 \longrightarrow \mathbb{C}$, $V = \mathbb{C}^2$, $W = \mathbb{C}$

  $A = \begin{pmatrix} 1 & i \end{pmatrix}$ $\quad \operatorname{Ker} A = ? \quad \operatorname{Im} A = ?$

  $$A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 1 z_1 + i z_2 = z_1 + i z_2$$

  $\longrightarrow$ Any complex number u can be reached, e.g. set $z_1 = u$, $z_2 = 0$.

  $\Rightarrow \operatorname{Im} A = \mathbb{C}$

  $$\overset{!}{=} 0 \Rightarrow z_1 = -i z_2$$

  $$\Rightarrow \operatorname{Ker} A = \left\{ \vec{v} \in \mathbb{C} \, ; \, \vec{v} = \begin{pmatrix} -i z_2 \\ z_2 \end{pmatrix} \right\}$$

  Of course, $\vec{0} \in \operatorname{Ker} A$ (take $z_2 = 0$), but also $\begin{pmatrix} -i \\ 1 \end{pmatrix} \in \operatorname{Ker} A$

  $\longrightarrow A^{-1}$ is not well-defined.

- $B: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

  $$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

  Ker B ? $\quad B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix} \overset{!}{=} 0 \Rightarrow \begin{matrix} x_1 = -2x_2 \\ x_1 = -\frac{4}{3} x_2 \end{matrix} \rightarrow x_1 = x_2 = 0$

  Im B? $\quad B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \longrightarrow$ solve the linear system for generic b_1, b_2.

  $$\det B = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

So B^{-1} is well defined. What is it?

$$B B^{-1} = \mathbb{1} \iff \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1)$$

$$\searrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2)$$

Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{4}{\det B}, \quad x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{-3}{\det B}$$

Repeat for (2)

$$\Rightarrow B^{-1} = \frac{1}{\det B} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

- $\frac{d}{dx} : \mathbb{V} \longrightarrow \mathbb{W}$ — spaces of functions

$$\frac{df}{dx} = -2x$$

$$A\vec{v} = \vec{b} \qquad f(x) = \int^x -2x' \, dx' + C = -x^2 + c$$

No well defined inverse! Any c gives $\frac{df}{dx} = -2x$.

We need to add another condition...

e.g. $f(0) = 1$ $\longleftarrow$ as part of the problem or the space $\mathbb{V}$.

Then, $f(x) = -x^2 + 1$

More generally,

$$\frac{df}{dx} = g(x) \Rightarrow f(x) = \int^x g(x') \, dx' + c$$

$$f(a) = f_0 \qquad c = f_0 - \int^a g(x') \, dx'$$