Short Takes 331

Linear operators : Part 1

Definition: A linear operator is a mapping A between to vector spaces $\mathbb{V}$, $\mathbb{W}$ that satisfies two properties:

① $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 \quad , \quad \vec{v}_1, \vec{v}_2 \in \mathbb{V}$

② $A(c\vec{v}) = cA\vec{v} \quad , \quad \vec{v} \in \mathbb{V}$

Examples

$\boxed{1}$ $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 2b \\ 3a + 4b \end{pmatrix} \in \mathbb{W}$$

$$A\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = A\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + 2(x_2 + y_2) \\ 3(x_1 + y_1) + 4(x_2 + y_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix} + \begin{pmatrix} y_1 + 2y_2 \\ 3y_1 + 4y_2 \end{pmatrix}$$

$$= A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

All matrices are, in fact, linear operators!
(more on this soon!)

$\boxed{2}$ $\dfrac{d}{dx}$

Indeed $\dfrac{d}{dx}(f+g) = \dfrac{df}{dx} + \dfrac{dg}{dx} \quad , \quad \dfrac{d(cf)}{dx} = c \cdot \dfrac{df}{dx}$

- Remember that vectors are abstract entities.

When we write $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ we are implicitly indicating the coordinates of $\vec{v}$ in the basis $\underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{\text{"Canonical basis"}}$, i.e.

$$\vec{v} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If we change our basis, we get a different representation for $\vec{v}$.

$$\vec{v} = 3 \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + -1 \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \rightarrow \vec{v} \rightarrow \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

in the $\left\{ \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \right\}$ basis.

The same holds for linear operators!

They are represented by matrices once we fix the basis.

For example

V = space generated by linear combinations of $\{1, x, x^2\}$

$\rightarrow$ vectors take the form $a + bx + cx^2$

$\rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ coordinates in this basis

$$A = \frac{d}{dx}, \quad A1 = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad Ax = 1 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A x^2 = 2x \rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \boxed{A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}$$

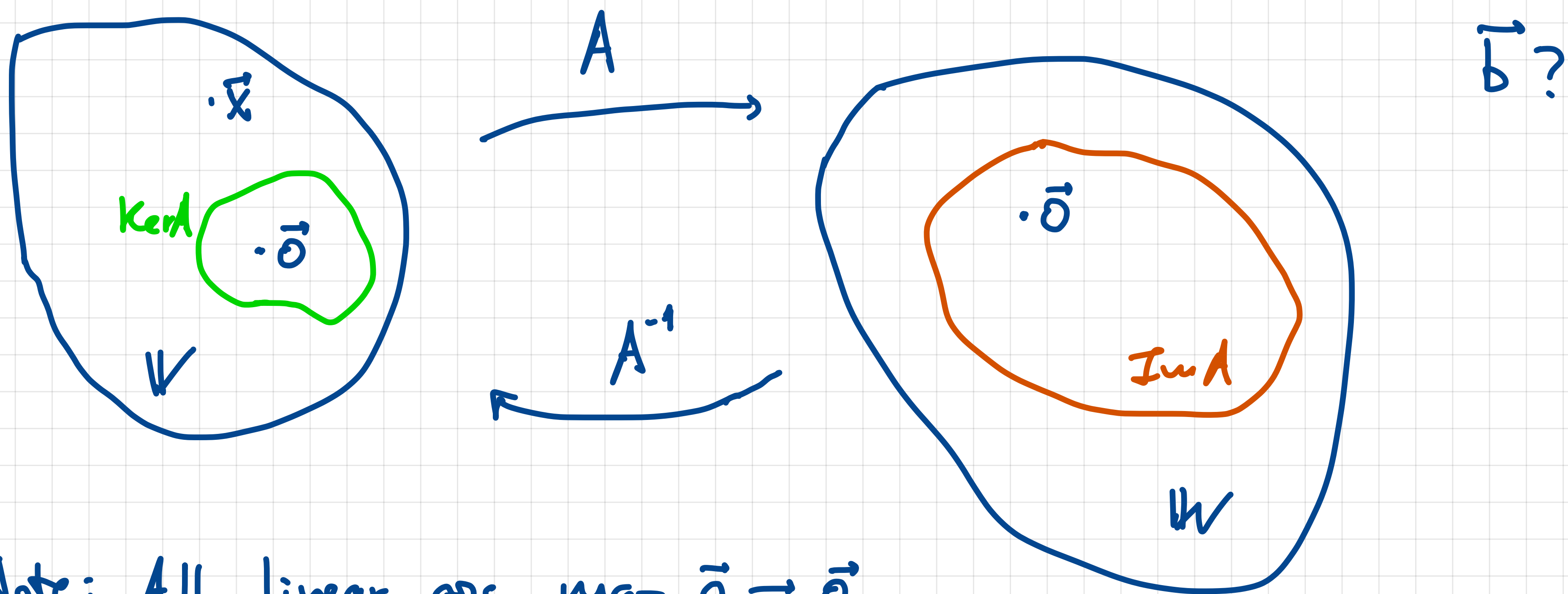
Makes sense!

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix} \rightarrow b + 2c\,x$$

$$\downarrow$$

$$a + bx + cx^2$$

- To solve problems $A\vec{x} = \vec{b}$, for a given A, it is useful to understand...
  - Does A^{-1} exist?
  - Is $\vec{b}$ in the set of possible results?
    What does this look like? Im A



- Note: All linear ops map $\vec{0} \rightarrow \vec{0}$

$$A\vec{x} = A(\vec{x} + \vec{0}) = A\vec{x} + A\vec{0} \Rightarrow A\vec{0} = \vec{0}$$

Ker A is the set of all $\vec{x}$ (including $\vec{x} = 0$), such that $A\vec{x} = 0$.