Short Takes 331

Linear systems of equations. Part 3.

$$\begin{cases} a x_1 + b x_2 = y_1 \\ c x_1 + d x_2 = y_2 \end{cases} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- How do we solve this in practice?

$$② \to d x_2 = y_2 - c x_1 \Rightarrow x_2 = \frac{y_2 - c x_1}{d}$$

$$① \to a x_1 + b\left(\frac{y_2 - c x_1}{d}\right) = y_1 \Rightarrow d a x_1 + b y_2 - b c x_1 = d y_1$$

$$\Rightarrow (da - bc) x_1 = d y_1 - b y_2$$

$$\Rightarrow \boxed{x_1 = \frac{d y_1 - b y_2}{da - bc}}$$

Similarly, $\boxed{x_2 = \frac{a y_2 - c y_1}{da - bc}}$

Note: - The solution is non-linear in the matrix coefficients (a, b, c, d)

- We will run into trouble if $da - bc = 0$!

If this is true, then

$$d a x_1 + d b x_2 = d y_1$$

and so

$$b c x_1 + d b x_2 = d y_1 \Rightarrow c x_1 + d x_2 = \frac{d y_1}{b}$$

. It looks like either...

1. We have no solution if $y_2 \neq \frac{d y_1}{b}$

2. We have only one eq. (infinite solutions) if $y_2 = \frac{d y_1}{b}$

- This peculiar quantity $da - bc$ is called a "determinant". It can be generalized to any matrix. We will do so later. Notation often used:

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \qquad (\text{for } 2\times 2 \text{ case})$$

When $\det A = 0$, we run into trouble as mentioned above.

- How do we solve the general (square) problem

$$A\vec{x} = \vec{b}$$

assuming we know $\det A \neq 0$?

One way to do it...

- Gaussian elimination

Example

$$\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

Form $\left(\begin{array}{cc|c} 3 & 4 & 7 \\ 5 & 2 & 7 \end{array}\right)$

① Take 1st row, divide by 3

$$\left(\begin{array}{cc|c} 3 & 4 & 7 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 4/3 & 7/3 \end{array}\right)$$

② Multiply by 5 & subtract from second row

$$\rightarrow \left(\begin{array}{cc|c} 5 & 20/3 & 35/3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 3 & 4 & 7 \\ 0 & 2 - 20/3 & 7 - 35/3 \end{array}\right) = \left(\begin{array}{cc|c} 3 & 4 & 7 \\ 0 & -14/3 & -14/3 \end{array}\right) \qquad \text{Triangular form!}$$

$$\Rightarrow \begin{cases} 3x_1 + 4x_2 = 7 \\ -\frac{14}{3}x_2 = -\frac{14}{3} \end{cases} \Rightarrow x_2 = 1 \Rightarrow x_1 = 1$$

Solved!

General case

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

0 Form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} & b_n \end{array}\right)$$

1 Divide first row by a_{11}

1.1 Multiply by a_{21} & subtract from 2nd row;

1.2 Multiply by a_{31} & subtract from 3rd row;

$\vdots$

1.n ... a_{n1} ... n-th row.

$$\longrightarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & a'_{22} & & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a'_{n2} & & a'_{nn} & b'_n \end{array}\right)$$

2 Divide second row by a'_{22}

2.1

2.2

$\vdots$

2.n

$$\longrightarrow \left(\begin{array}{ccccc|c} a_{11} & a_{12} & \dots & & a_{1n} & b_1 \\ 0 & a'_{22} & \dots & & a'_{2n} & b'_2 \\ \vdots & 0 & a''_{33} & \dots & \vdots & \vdots \\ 0 & 0 & a''_{n3} & \dots & a''_{nn} & b''_n \end{array}\right)$$

③ Repeat this process until you obtain an upper triangular system.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ & \vdots & 0 & & \\ 0 & 0 & 0 & a^{'''\cdots}_{nn} & b^{'''\cdots}_n \end{array}\right)$$

④ Solve by backsubstitution.