Short Takes 331

Linear systems of equations. Part 2.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Using matrices and vectors, we can rewrite it as

$$\boxed{A\vec{x} = \vec{b}}$$

Our job is to solve for $\vec{x}$.

$m > n$: more equs. than unknowns $\rightarrow$ no solution

$m = n$: as many equs. as unknowns $\rightarrow$ unique solution

$m < n$: fewer equs. than unknowns $\rightarrow$ $\underbrace{\text{infinitely many solutions}}_{\text{"typically"}}$

Caveats: degeneracy & consistency

degeneracy $\downarrow$ equs. contain "superfluous" information

E.g.

$$\begin{cases} x_1 + 2x_2 = 9 \\ -2x_1 - 4x_2 = -18 \end{cases}$$

Second eq. is -2 times the first.
$\hookrightarrow$ they are linearly dependent!

consistency $\hookrightarrow$ equs. contain "inconsistent" information

E.g.

$$\begin{cases} 2x_1 + 3x_2 = -1 \\ -2x_1 - 3x_2 = 7 \end{cases}$$

$\hookrightarrow 0 = 6$ ⊗ ???

- Solving the problem amounts to putting A on the other side of the equation:

$$A\vec{x} = \vec{b} \longrightarrow A^{-1}A\,\vec{x} = A^{-1}b \longrightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$$

$$\downarrow$$
$$A^{-1}A = \mathbb{1}$$

If A^{-1} exists, the above operations are allowed and the solution vector $\vec{x}$ is unique.

- How do we know when A^{-1} exists?
- How can one construct it?
- Is there a way to find $\vec{x}$ directly without going through finding A^{-1}?

<u>Note</u>: Knowing A^{-1} allows us to solve for $\vec{x}$ for <u>any</u> given $\vec{b}$.

Therefore, knowing $\vec{x}$ for a particular $\vec{b}$ is a lot less information.

To gain intuition, let's look at some simple cases...

<u>Case 1:</u>

$$A = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$n \times n$

$$A\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \longrightarrow \text{trivial!}$$

$$\boxed{\begin{aligned} x_1 &= b_1 \\ x_2 &= b_2 \\ &\vdots \end{aligned}}$$

<u>Case 2:</u>

$$A = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \longrightarrow \text{still rather trivial...}$$

$$\boxed{\begin{aligned} x_1 &= b_1/d_1 \\ x_2 &= b_2/d_2 \\ &\vdots \end{aligned}}$$

<u>must have</u> $d_k \neq 0 \;\; \forall k$

- what if some $d_k = 0$?
  ↳ the corresponding x_k are <u>arbitrary</u> & the solution is <u>not unique</u>, if $b_k = 0$.

  If $b_k \neq 0$, there is a contradiction ⟶ <u>no solution</u>.
  (Consistency!)

Case 3:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & 0 & a_{33} & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & & a_{nn} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$A\vec{x} = ?$ "upper triangular"

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{22}x_2 + \dots + a_{2n}x_n \\ \ddots \quad \vdots \\ a_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$\Rightarrow$

* $\boxed{x_n = b_n / a_{nn}}$ (assuming $a_{nn} \neq 0$)

* $a_{n-1,n-1}\, x_{n-1} + a_{n-1,n}\, \underline{x_n} = b_{n-1}$

  known from previous eqn. $\rightarrow$. use it to solve for x_{n-1}. . will need $\underline{a_{n-1,n-1} \neq 0}$.

* The next eqn. will involve x_{n-2}, x_{n-1}, x_n, but we already know the last two, so we use them to find x_{n-2}.

  That will require $\underline{a_{n-2,n-2} \neq 0}$.

* Continuing in this fashion we solve <u>for all x_k</u>. We will end up requiring $\underline{a_{kk} \neq 0}$, i.e. must have non-zero diagonal elements.

Lower triangular systems are solved in the same way, but starting from the top.