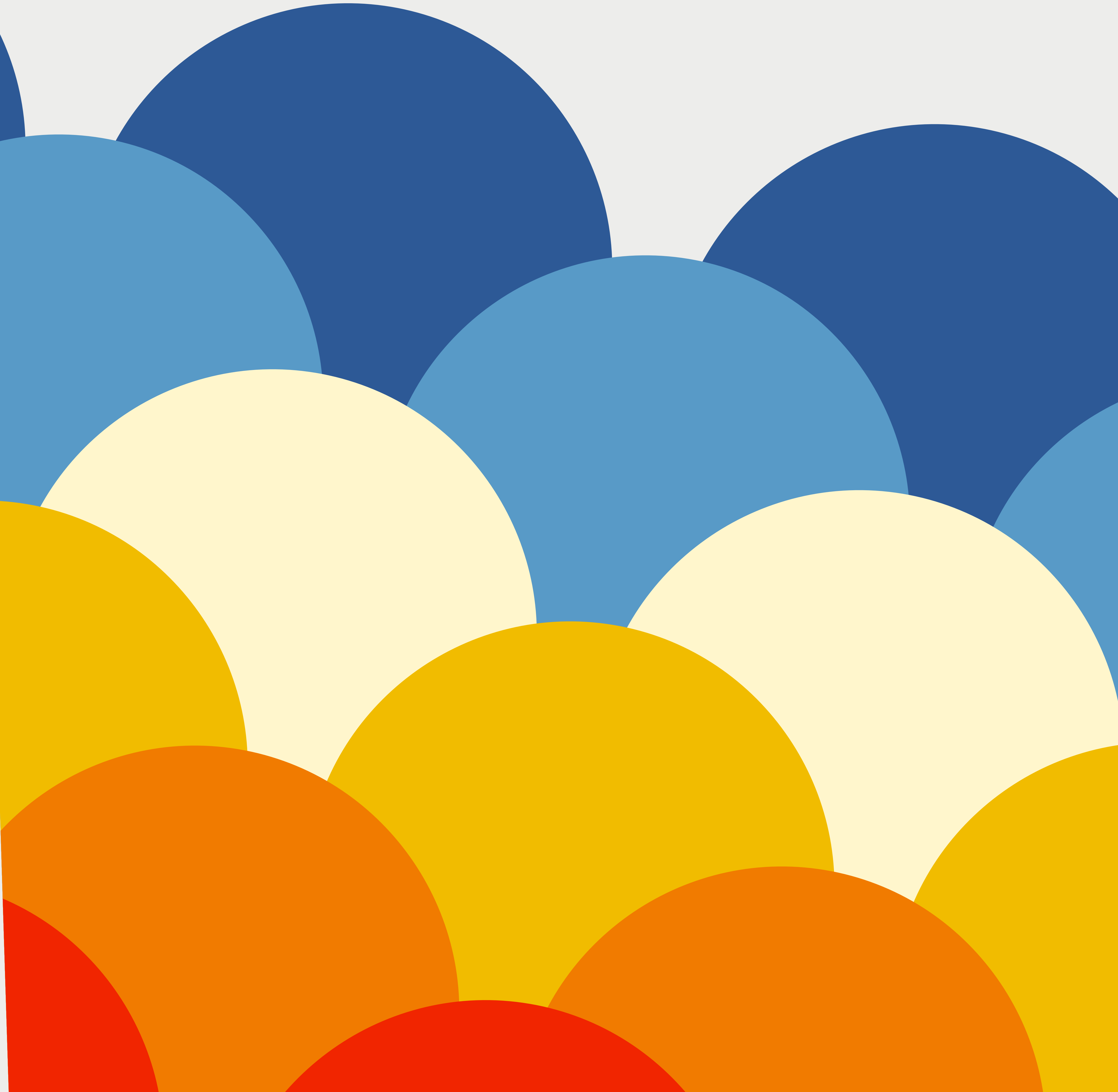
Short Takes 331



Linear systems of equations. Part 1.

Examples:
$$\begin{cases} x_1 + 3x_2 = 9 \\ x_1 - 7x_2 = 1 \end{cases}$$
unknowns: x_1, x_2

$$\begin{cases} 7x + \pi . y - iz = 0 \\ \quad\quad y + z = 21 \\ x - 2y + 3z = 1 \end{cases}$$
unknowns: x, y, z

The unknowns always appear linearly (power 1).

Using matrices and vectors, we can rewrite the above as

$$\boxed{A\vec{x} = \vec{b}}$$

with $A = \begin{pmatrix} 1 & 3 \\ 1 & -7 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$

and $A = \begin{pmatrix} 7 & \pi & -i \\ 0 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 21 \\ 1 \end{pmatrix}$

Our job is to solve for $\vec{x}$.

We may have more generally

$$A \in \mathbb{F}^{m \times n} \qquad \vec{x} \in \mathbb{F}^n \qquad \vec{b} \in \mathbb{F}^m$$

Three cases:

1. $m > n$: more equations than unknowns $\to$ no solution (unless there are superfluous eqns)

2. $m = n$: as many eqns as unknowns
$\to$ it is possible to have a unique solution (unless superfluous eqns $\to$ infinite solutions)

3 $m < n$: fewer eqns than unknowns
$\rightarrow$ infinite number of solutions

1_ $m > n$

- $$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2/3 \end{pmatrix}$$ $\rightarrow$ too many constraints!
no solution

- $$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$ $\leftarrow$ last eqn is superfluous!

2_ $m = n$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= -1 \\ 2x_1 + 4x_2 &= 0 \end{aligned} \Rightarrow x_1 = -2x_2 \Rightarrow \begin{cases} x_2 = -1 \\ x_1 = 2 \end{cases} \checkmark$$

$\rightarrow$ unique solution!

3_ $m < n$

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \end{pmatrix}$$

$$x_1 + 3x_2 = -1 \Rightarrow x_2 = \frac{-1 - x_1}{3}, \quad x_1 \text{ anything}$$

$\rightarrow$ infinite number of possible solutions.

- In the above, we assumed $\vec{b} \neq 0$ → "inhomogeneous systems of eqns"

If $\vec{b} = 0$, then there is always a solution, namely $\vec{x} = 0$.

Indeed,
$$\underbrace{A\vec{x} = 0}_{\text{"homogeneous system of equations"}} \quad \text{if } \vec{x} = 0, \text{ no matter the shape of } A.$$

- Examples

  - $$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

    Clearly $x_1 = x_2 = 0$ is a solution.

    other solutions? $x_1 + 3x_2 = 0 \Rightarrow x_1 = -3x_2$

    $2x_1 + 4x_2 = -6x_2 + 4x_2 = -2x_2 = 0$

    $\Rightarrow x_2 = 0, \; x_1 = 0$

    No other solutions!

  - $$\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \Rightarrow x_1 = -3x_2, \quad x_2 \text{ arbitrary}$$

    "under-determined"

    infinitely many solutions

  - $$\begin{pmatrix} 1 & 3 & 2 \\ -2 & -6 & -4 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = 0$$

    $x_1 + 3x_2 + 2x_3 = 0$

    $\Rightarrow x_2 = \frac{2}{3} x_3$