Short Takes 331

Special matrices & matrix properties

- The product of two matrices A & B is not in general commutative: $AB \neq BA$ in general.

Example $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

then

$$AB = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

whereas

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

- If $D = \begin{pmatrix} d_1 & & & \emptyset \\ & d_2 & & \\ & & \ddots & \\ \emptyset & & & d_n \end{pmatrix}$, D is "diagonal"

E.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ also $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

"proportional to 1"

- The product of diagonal matrices is always commutative.

$$DC = CD = \begin{pmatrix} d_1c_1 & & & \emptyset \\ & d_2c_2 & & \\ & & \ddots & \\ \emptyset & & & d_nc_n \end{pmatrix} = \begin{pmatrix} c_1d_1 & & & \emptyset \\ & c_2d_2 & & \\ & & \ddots & \\ \emptyset & & & c_nd_n \end{pmatrix}$$

- If given B, $\exists A$ such that $AB = BA = I$, then B is said to be "invertible" and $A = B^{-1}$.

→ In general, finding A^{-1} for a given A is a difficult problem and there are cases where there is no solution (more in future videos!). However...

Examples

- $I^{-1} = I$

- D diagonal w/ non-zero diagonal elements
$$D = \begin{pmatrix} d_1 & & & \emptyset \\ & d_2 & & \\ & & \ddots & \\ \emptyset & & & d_n \end{pmatrix}$$
then
$$D^{-1} = \begin{pmatrix} d_1^{-1} & & & \emptyset \\ & d_2^{-1} & & \\ & & \ddots & \\ \emptyset & & & d_n^{-1} \end{pmatrix}$$

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (verify!)

- $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \leftarrow$ not invertible!

- If $U = \begin{pmatrix} u_{11} & u_{21} & \cdots & \\ & u_{22} & \cdots & \vdots \\ \emptyset & & \ddots & u_{nn} \end{pmatrix}$, U is "upper triangular" (non-zeros on diagonal and above)

Similarly, we define "lower triangular" matrices as
$$L = \begin{pmatrix} L_{11} & & & \emptyset \\ L_{12} & L_{22} & & \\ \vdots & \vdots & \ddots & \\ & & & L_{nn} \end{pmatrix}$$

- Given M, of size $m \times n$
  - M^T "M transpose" ; $[M^T]_{ij} = M_{ji}$
  - M^* "M complex conjugate" ; $[M^*]_{ij} = (M_{ij})^*$
    (sometimes $\overline{M}$)
  - $M^\dagger$ "M dagger" or "hermitian conjugate" ; $M^\dagger = (M^T)^*$
- With these definitions, we further define ...

  If ...
  - $A = A^T \rightarrow A$ is "symmetric"

    E.g. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, any a, b, c
  - $A = -A^T \rightarrow A$ is "anti-symmetric"

    E.g. $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}$

    Notice: . Signs
    . diagonal
  - By the way ... for any square matrix M,

    $$M = \frac{M}{2} + \frac{M}{2} = \underbrace{\frac{M + M^T}{2}}_{\text{symmetric}} + \underbrace{\frac{M - M^T}{2}}_{\text{antisymmetric}}$$

- $H = H^\dagger \longrightarrow H$ is "Hermitian"

E.g. $\begin{pmatrix} 1 & z & 0 \\ z^* & 3 & i \\ 0 & -i & -5 \end{pmatrix}$, $z \in \mathbb{C}$

$a+ib$, $a, b \in \mathbb{R}$

Notice: . signs
. diagonal

- $V^T = V^{-1} \longrightarrow V$ is "orthogonal"

E.g. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ usually represent rotations

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- $U^\dagger = U^{-1} \longrightarrow U$ is "unitary"

E.g. $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ used a lot in QM!

This is one of the Pauli matrices. It is also Hermitian.