Short Takes 331

Matrix-vector and matrix-matrix multiplication

Matrix: An array of numbers, e.g.

$$M = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & \\ \vdots & & \ddots & \\ & & & M_{mn} \end{pmatrix}$$

$m \times n$

rows columns

If M is made out of real numbers, $M \in \mathbb{R}^{m \times n}$

Given a vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, we define the product $M\vec{x}$ by

$$M\vec{x} = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & \ddots & & \\ \vdots & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} M_{11} x_1 + M_{12} x_2 + \dots + M_{1n} x_n \\ M_{21} x_1 + M_{22} x_2 + \dots + M_{2n} x_n \\ \vdots \\ M_{m1} x_1 + M_{m2} x_2 + \dots + M_{mn} x_n \end{pmatrix}$$

For this to work, the number of columns of M must be the same as the number of rows of $\vec{x}$.

(m rows $\times$ n columns) $\times$ (n rows) = (m rows)
matrix vector vector

- Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$$

This is very useful to represent linear equations, e.g.

$$\begin{cases} x_1 + 2x_2 = 7 \\ 3x_1 + 4x_2 = -2 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

Notice that we may also write $M\vec{x}$ as

$$M\vec{x} = \begin{pmatrix} \vec{m}_1 & \vec{m}_2 & \vec{m}_3 & \dots & \vec{m}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{x_1 \vec{m}_1 + x_2 \vec{m}_2 + \dots + x_n \vec{m}_n}$$

$\vec{m}_k$ = k-th column of M

Very useful! It shows that the resulting vector is just a linear combination of the columns of M.

• Example:

$$\begin{pmatrix} 7 & 1 \\ 10 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7\cdot 2 + 1\cdot 3 \\ 10\cdot 2 + (-1)\cdot 3 \end{pmatrix} = \begin{pmatrix} 17 \\ 17 \end{pmatrix}$$

or $\begin{pmatrix} 7 \\ 10 \end{pmatrix} \cdot 2 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot 3 = \begin{pmatrix} 14 \\ 20 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 17 \\ 17 \end{pmatrix}$

• Example: Unit matrix $\mathbb{I}$ (or $\mathbb{1}$)

$$\mathbb{I}_{ij} = \delta_{ij} \Rightarrow \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1\times(-2) + 0\times 5 + 0\times 8 \\ 0\times(-2) + 1\times 5 + 0\times 8 \\ 0\times(-2) + 0\times 5 + 1\times 8 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 8 \end{pmatrix}$$

Same vector we started with!

Matrix-matrix multiplication

$A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

Define AB by

$$[AB]_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \qquad \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, p \end{array}$$

AB is a matrix of size $m \times p$

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{m1} & \dots & & A_{mn} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1p} \\ \vdots & & & \\ B_{n1} & B_{n2} & \dots & B_{np} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} A_{1k} B_{k1} & \dots \\ \vdots & \ddots \\ & \end{pmatrix}$$

- Example:

$$\begin{pmatrix} 7 & 1 \\ 10 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 7 \times 2 + 1 \times 3 & 7 \times 1 + 1 \times 5 \\ 10 \times 2 + (-1) \times 3 & 10 \times 1 + (-1) \times 5 \end{pmatrix} = \begin{pmatrix} 17 & 12 \\ 17 & 5 \end{pmatrix}$$

Try $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ 10 & -1 \end{pmatrix}$

- Example:

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix}$$