Short Takes 331

Vector coordinates & change of basis.

So far,

- vectors are the elements of a vector space $\mathbb{V}$.
- Examples:
  - $(1,0,0)$, $(3.7, \pi, -1)$, etc
  - $1, x, x^2, \sin x, \ldots$

Given a vector $\vec{v} \in \mathbb{V}$ and a basis $X = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_N\}$, there will be constants a_i such that

$$\vec{v} = \sum_{i=1}^{N} a_i \vec{w}_i$$

basis vector

our vector

"coordinates of $\vec{v}$ in the X basis"

but what if we chose a different basis? $Y = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N\}$

Then,

$$\vec{v} = \sum_{j=1}^{N} b_j \vec{u}_j$$

"coordinates of $\vec{v}$ in the Y basis"

For a given problem, some coordinates may be more convenient than others to represent $\vec{v}$...

... but the vector $\vec{v}$ itself remains the same!

It has a life of its own, regardless of how we choose to describe it.

We refer to $\vec{v}$ as an "abstract vector".

Since there are different bases to choose from, we may want to switch from one to another.

How do we do that?

Take the basis Y above and note that

$$\boxed{\vec{u}_j = \sum_{i=1}^{N} T_{ji}\, \vec{w}_i}$$

$j = 1, \ldots, N$

coordinates of $\vec{u}_j$ in the X basis

Then, for any $\vec{v}$,

$$\vec{v} = \sum_{j=1}^{N} b_j \vec{u}_j = \sum_{j=1}^{N} b_j \left(\sum_{i=1}^{N} T_{ji}\, \vec{w}_i \right)$$

$$= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} b_j T_{ji} \right) \vec{w}_i$$

but we know that, in the X basis,

$$\vec{v} = \sum_{i=1}^{N} a_i \vec{w}_i \quad \Rightarrow \quad \sum_{i=1}^{N} \underbrace{\left[a_i - \left(\sum_{j=1}^{N} b_j T_{ji} \right) \right]}_{\to 0} \vec{w}_i = 0 .$$

Since the $\vec{w}_i$ are LI, the only way the above is possible is if, for each i,

$$\boxed{a_i = \sum_{j=1}^{N} b_j T_{ji}}$$

$i = 1, \ldots, N$

The coefficients T_{ji} that connect X with Y, serve to transform the coefficients from the b coords to the a coords.

If we arrange $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & \cdots \\ T_{21} & T_{22} & \cdots & \\ \vdots & & \ddots & \end{pmatrix}$ ← "matrix"

more on these soon!

Since all bases are in principle valid, the reverse operation also exists, i.e.

$$\boxed{b_k = \sum_{i=1}^{N} a_i U_{ik}}$$

$k = 1, \ldots, N$

for certain coefficients U_{ij} which satisfy

$$\boxed{\vec{w}_i = \sum_{j=1}^{N} U_{ij} \vec{u}_j}$$

but then

$$b_k = \sum_{i=1}^{N} \sum_{j=1}^{N} b_j T_{ji} U_{ik}$$

$$= \sum_{j=1}^{N} b_j \underbrace{\left(\sum_{i=1}^{N} T_{ji} U_{ik} \right)}_{?}$$

for any coordinates b_k!

It must be that $\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ "Kronecker delta"

$$\Rightarrow \boxed{\sum_{i=1}^{N} T_{ji} U_{ik} = \delta_{jk}}$$

$$T\,U = \mathbb{1}$$

$U = T^{-1}$

"inverse matrix"

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$ "unit matrix"

Example: $\mathbb{V} = \mathbb{R}^2$

$$X = \{\vec{w}_1, \vec{w}_2\}, \quad \vec{w}_1 = (2,0), \quad \vec{w}_2 = (1,3)$$

$$Y = \{\vec{u}_1, \vec{u}_2\}, \quad \vec{u}_1 = (1,1), \quad \vec{u}_2 = (1,-1)$$

- $\vec{u}_1 = T_{11}\vec{w}_1 + T_{12}\vec{w}_2 = (2T_{11}, 0) + (T_{12}, 3T_{12})$
  $= (2T_{11} + T_{12}, 3T_{12}) = (1,1)$

$$\Rightarrow \begin{cases} T_{12} = 1/3 \\ T_{11} = (1 - 1/3)\frac{1}{2} = 1/2 - 1/6 = \frac{1}{3} \end{cases}$$

- $\vec{u}_2 = T_{21}\vec{w}_1 + T_{22}\vec{w}_2 = (2T_{21} + T_{22}, 3T_{22}) = (1,-1)$

$$\Rightarrow \begin{cases} T_{22} = -1/3 \\ T_{21} = (1 + 1/3)\frac{1}{2} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \end{cases}$$

Therefore,
$$T = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}$$

Exercise: Find the matrix U corresponding to the inverse operation.

Example: $X = \{1, x, x^2\} \qquad Y = \{3, x-1, 2x^2 - x + 7\}$

$$3 = 3.1 \longrightarrow T_{11} = 3, \quad T_{12} = 0, \quad T_{13} = 0$$

$$x - 1 = 1.x + (-1).1 \longrightarrow T_{21} = -1, \quad T_{22} = 1, \quad T_{23} = 0$$

$$2x^2 - x + 7 = 2.x^2 + (-1)x + 7.1 \longrightarrow T_{31} = 2, \quad T_{32} = -1, \quad T_{33} = 7$$