Short Takes 331

Gaussian integrals & Feynman's trick

In a previous video we used that

$$I_0(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx = \sqrt{\frac{\pi}{\lambda}} \qquad , \quad \lambda > 0$$

and argued that we can use this result to calculate

$$I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx \,,$$

Indeed, we saw that

$$\frac{\partial}{\partial\lambda} I_0(\lambda) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\lambda}\left(e^{-\lambda x^2}\right) dx = \int_{-\infty}^{\infty} (-x^2)\, e^{-\lambda x^2}\, dx$$

$$\Rightarrow \quad -\frac{\partial}{\partial\lambda} I_0(\lambda)\Big|_{\lambda=1} = I_1$$

$$= \frac{\sqrt{\pi}}{2}$$

- We could take higher derivatives of $I_0(\lambda) = \sqrt{\frac{\pi}{\lambda}}$ to generate all the I_n we want:

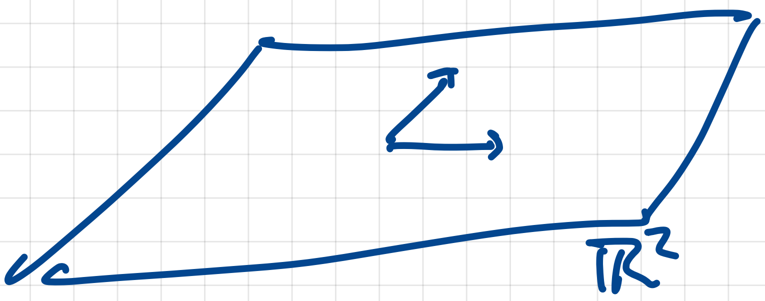
$$I_n = (-1)^n \left[\frac{\partial^n I_0(\lambda)}{\partial\lambda^n}\right]\Bigg|_{\lambda=1}$$

- **Main problem**: how do we calculate $I_0(\lambda)$?

$$\boxed{I_0(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx} \underset{t=\sqrt{\lambda}x}{=} \boxed{\int_{-\infty}^{\infty} e^{-t^2}\, dt \cdot \frac{1}{\sqrt{\lambda}}}$$

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \ ?$$

$\longrightarrow$ Calculate
$$\left(\int_{-\infty}^{\infty} e^{-t^2}\, dt\right)\left(\int_{-\infty}^{\infty} e^{-v^2}\, dv\right) = \int_{\mathbb{R}^2} e^{-(t^2+v^2)}\, dt\, dv =$$



$$t^2 + v^2 = r^2$$

$$dt\, dv = r\, d\varphi$$

$$\varphi \in [0, 2\pi)$$
$$r \in [0, \infty)$$

$$= \int_{\mathbb{R}^2} e^{-r^2}\, r\, dr\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\varphi = \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\ \underbrace{\int_0^{\infty} e^{-r^2}\, r\, dr}_{1/2} = \pi$$

Therefore,

$$\boxed{\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}} \quad \text{and so} \quad \boxed{I_0(\lambda) = \sqrt{\frac{\pi}{\lambda}}}$$

- Feynman's trick in other cases.

Two principles:

1. We must be able to insert a parameter in a clever spot to generate the integrals we want.

2. The "base case" ($I_0(\lambda)$ above) must be an integral we can solve analytically for all values of λ.

Example: $J_n = \int_0^1 (\ln x)^n \, x^2 \, dx$

→ we could solve it by $y = \ln x \rightarrow x^2 = e^{2y}$

$x = e^y \qquad dx = e^y dy$

such that

$$J_n = \int_{-\infty}^{0} y^n e^{3y} \, dy$$

which we can do with usual methods.

→ But in this case Feynman provides a very elegant path

Take $x^2 = e^{2 \ln x}$

so we may define

$$\boxed{J_0(\lambda)} = \int_0^1 x^\lambda \, dx = \int_0^1 e^{\lambda \ln x} \, dx$$

$$= \frac{1}{\lambda+1} x^{\lambda+1} \Big|_0^1 = \boxed{\frac{1}{\lambda+1}}$$

Then, $\frac{\partial}{\partial \lambda} J_0(\lambda) = \int_0^1 (\ln x) \, x^\lambda \, dx = -\frac{1}{(\lambda+1)^2}$

$$\xrightarrow[\lambda \to 2]{} J_1$$

and so on with n derivatives for J_n.

Exercise: Generalize the above to

$$J_{nm} = \int_0^1 (\ln x)^n \, x^m \, dx$$