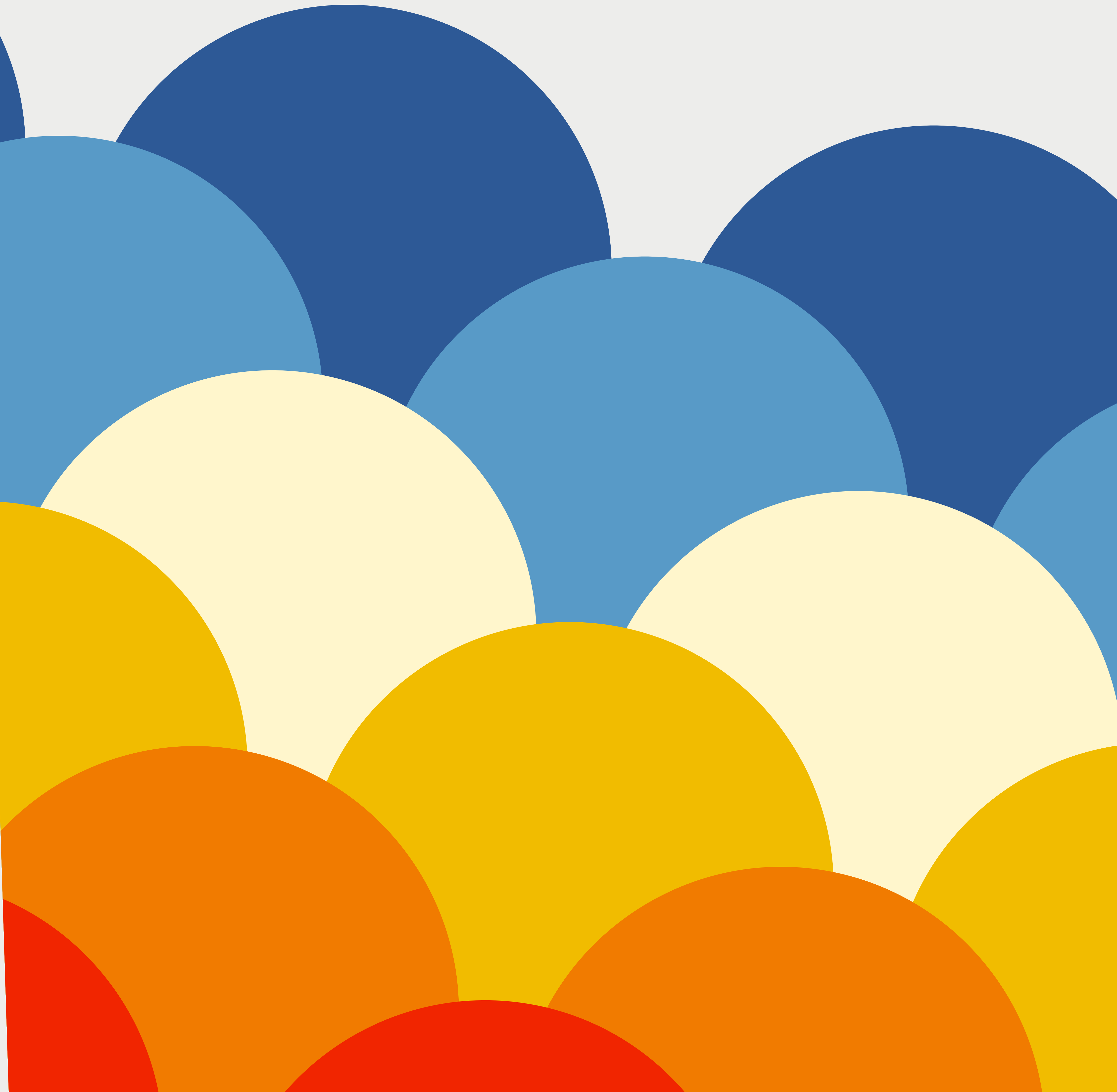
Short Takes 331



Orthogonal functions & special polynomials

Recall that given

$$X = \{1, x, x^2, \dots\} \quad , \quad \text{which are LI},$$

we may orthogonalize (and optionally also normalize) these monomials using the Gram-Schmidt process.

For that purpose, we need in our space of functions to have a well-defined inner product.

For example,

A $(f,g) = \int_{-1}^{1} f(x)g(x)\,dx$ or B $(f,g) = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx$

or perhaps

C $(f,g) = \int_{0}^{\infty} f(x)g(x)e^{-x}\,dx$, and so on.

- Key point: the form of the inner product will determine the type of polynomials we end up with upon orthogonalization

- If we use A, then Gram-Schmidt will produce the Legendre polynomials, whereas B will produce the Hermite polynomials!

  You too can come up with your own inner product and invent your own "You" polynomials!

In a previous video we did the first 3 cases of A

Let's try B!

- Recall GS ... $\quad v_1 = 1, \quad v_2 = x, \quad v_3 = x^2, \ldots$

$$(f,g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-x^2}\, dx$$

$$\boxed{w_1 = 1}$$

$$\boxed{w_2} = v_2 - \frac{(v_2, w_1)}{(w_1, w_1)} \cdot w_1 = x - \frac{\int_{-\infty}^{\infty} x.1.e^{-x^2} dx}{\int_{-\infty}^{\infty} 1.1.\, e^{-x^2} dx} \; 1$$

but
$$\int_{-\infty}^{\infty} \underbrace{x}_{\text{odd function}} \underbrace{e^{-x^2}}_{\text{even function}} dx = 0$$

$$= \boxed{x}$$

$$w_3 = v_3 - \frac{(v_3, w_1)}{(w_1, w_1)} w_1 - \frac{(v_3, w_2)}{(w_2, w_2)} w_2$$

$$(v_3, w_1) = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \, ?$$

$$(v_3, w_2) = \int_{-\infty}^{\infty} x^3 e^{-x^2} dx = 0$$

- We'll need

$$I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx$$

- We know

$$I_0(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}$$

$\rightarrow$ Feynman trick: $-\frac{\partial I_0(\lambda)}{\partial \lambda}\Big|_{\lambda=1} = I_1$

$$\frac{\partial}{\partial\lambda} I_0(\lambda) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\lambda}\left(e^{-\lambda x^2}\right) dx = \int_{-\infty}^{\infty} (-x^2)\, e^{-\lambda x^2} dx$$

$$\Rightarrow \quad -\frac{\partial}{\partial\lambda} I_0(\lambda)\Big|_{\lambda=1} = I_1$$

So far so good... but we also know $I_0(\lambda) = \sqrt{\frac{\pi}{\lambda}}$

Therefore,
$$-\frac{\partial}{\partial\lambda} I_0(\lambda) = -\sqrt{\pi}\left(-\frac{1}{2}\right)\lambda^{-3/2} = \frac{\sqrt{\pi}}{2}\cdot\lambda^{-3/2} \xrightarrow[\lambda\to 1]{} \frac{\sqrt{\pi}}{2} = I_1$$

Note:
- Taking more derivatives, we can access I_n for arbitrary n.
- By evaluating at different values of λ, we also get other integrals for free, such as
$$\int_{-\infty}^{\infty} x^2 e^{-2x^2} dx = \frac{\sqrt{\pi}}{2}\cdot 2^{-3/2}$$
- Knowing these Gaussian integrals is essential in Physics, not just in generating orthogonal polynomials.

Going back to w_3...

$$\boxed{w_3 = x^2 - \frac{(v_3, w_1)}{(w_1, w_1)} w_1 = x^2 - \frac{\sqrt{\pi}/2}{\sqrt{\pi}} = x^2 - 1/2}$$

Continuing in this way, we generate the Hermite polynomials.

- These are important functions because they characterize the quantum harmonic oscillator.
$$\psi \sim N e^{-x^2/2} H(c.x)$$

- Another case: Jacobi polynomials
$$(f, g) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\, g(x)\, dx \quad , \quad \alpha, \beta > -1 .$$