Short Takes 331

Orthogonal bases

Vector space $\mathbb{V}$ with an inner product $(\vec{v}, \vec{w})$, i.e. inner-product space.

$X = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N\}$ basis $\rightarrow$ LI and spans $\mathbb{V}$

$\dim \mathbb{V} = N$

X is an orthogonal basis iff

$(\vec{v}_i, \vec{v}_j) = 0$ if $i \neq j$, $i, j = 1, 2, \dots, N$

- Note: if $(\vec{v}_i, \vec{v}_i) = 1$, we say the basis is orthonormal (since vectors are normalized to length 1)

  and write $(\vec{v}_i, \vec{v}_j) = \delta_{ij}$

  Kronecker delta

  $$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

- Examples
  - $\mathbb{R}^3$ with conventional dot product

    $X = \{\breve{i}, \breve{j}, \breve{k}\}$ is an orthonormal basis

    $\breve{i} = (1, 0, 0)$ $\breve{j} = (0, 1, 0)$ $\breve{k} = (0, 0, 1)$

  - $\mathbb{R}^2$ $X = \{\breve{i}, \breve{j}\}$ or $\breve{e}_1 = (1/\sqrt{2}, 1/\sqrt{2})$

    $\breve{e}_2 = (1/\sqrt{2}, -1/\sqrt{2})$

- **Why define orthogonal bases?**

They are very useful!

Any vector $\vec{V}$ will take the form

$$\vec{V} = a_1 \vec{V}_1 + a_2 \vec{V}_2 + \dots + a_N \vec{V}_N$$

where we note that

$$\rightarrow \quad (\vec{V}, \vec{V}_1) = a_1 (\vec{V}_1, \vec{V}_1) + a_2 \underbrace{(\vec{V}_2, \vec{V}_1)}_{\to 0} + \dots + a_N \underbrace{(\vec{V}_N, \vec{V}_1)}_{\to 0}$$

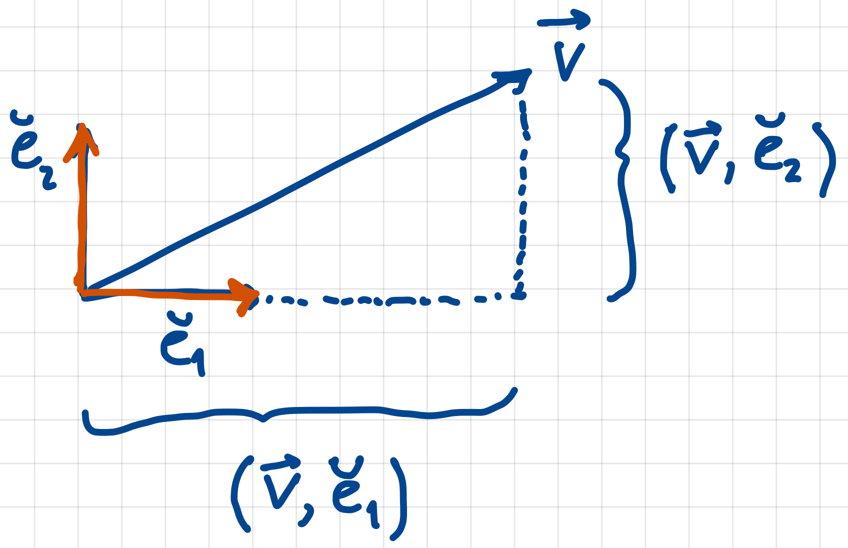
$$= a_1 (\vec{V}_1, \vec{V}_1)$$

$$= a_1 \quad \left[\text{if } (\vec{V}_1, \vec{V}_1) = 1 \ \ (\text{assume normalized!})\right]$$

Thus, the inner product allows us to calculate the coefficient a_1. We may then write, generally,

$$\vec{V} = \underbrace{(\vec{V}, \vec{V}_1)}_{\substack{\text{Projection of } \vec{V} \text{ in} \\ \text{the direction of } \vec{V}_1}} \vec{V}_1 + (\vec{V}, \vec{V}_2) \vec{V}_2 + \dots + (\vec{V}, \vec{V}_N) \vec{V}_N$$

Geometrically,



Algebraically,

$$\vec{V} = (7, 3)$$

$$\vec{V} = a_1 \breve{e}_1 + a_2 \breve{e}_2$$

$$a_1 = (\vec{V}, \breve{e}_1) = \dots = 10/\sqrt{2}$$

$$a_2 = (\vec{V}, \breve{e}_2) = \dots = 4/\sqrt{2}$$

- Examples in spaces of functions.

1 – Take $\mathbb{V}$ to be the space of all linear combinations of $\{1, x, x^2, x^3\}$

with the inner product

$$(f, g) = \int_{-1}^{1} f(x) g(x)\, dx$$

Clearly, $\displaystyle\int_{-1}^{1} 1 \cdot x\, dx = \int_{-1}^{1} x\, dx = \frac{1}{2} x^2 \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

and $\displaystyle\int_{-1}^{1} x \cdot x^2\, dx = \int_{-1}^{1} x^3\, dx = \ldots = 0$

However, $\displaystyle\int_{-1}^{1} 1 \cdot 1\, dx = 2$, $\displaystyle\int_{-1}^{1} x \cdot x\, dx = \frac{2}{3}$

and $\displaystyle\int_{-1}^{1} x \cdot x^3 \neq 0$

$= \frac{2}{5}$ $\longrightarrow$ it is not orthogonal!

We may normalize

$1 \longrightarrow \frac{1}{\sqrt{2}}$, $x \longrightarrow x / \sqrt{2/3}$, ...

but the basis will remain not orthogonal.

What if we want an orthogonal basis in this space? Can we find one?

Yes! $\longrightarrow$ Gram-Schmidt process

(future video!)

2 - Consider $\{\sin t, \sin 2t, \sin 3t, \dots\}$

with $$(f, g) = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

Then $$\int_{-\pi}^{\pi} \sin(nt) \sin(mt)\, dt = 0 \quad \text{if} \quad m \neq n.$$

They are orthogonal !

These functions are part of the Fourier basis, which is essential for Fourier series, as we will see later.