Short Takes 331

Inner product (a.k.a. "scalar product")

Given a vector space $\mathbb{V}$, an inner product is an operation involving two vectors (i.e. a "binary" operation) which associates a scalar to them...

$$(\vec{v}, \vec{w}) = s$$

$\vec{v}, \vec{w} \in \mathbb{V}$ are vectors

$s \in \mathbb{S}$ is a scalar

Notation: $\vec{v} \cdot \vec{w}$

$\langle \vec{v} | \vec{w} \rangle$ also common

... such that:

1 – $(\vec{v}, \vec{w}) = (\vec{w}, \vec{v})^*$ "Conjugate property"

2 – $(x\vec{v}, \vec{w}) = x(\vec{v}, \vec{w})$, $x \in \mathbb{S}$ a scalar

$(\vec{u} + \vec{v}, \vec{w}) = (\vec{u}, \vec{w}) + (\vec{v}, \vec{w})$ "linearity"

3 – $(\vec{v}, \vec{v}) \geqslant 0$

if $(\vec{v}, \vec{v}) = 0$ then $\vec{v} \equiv \vec{0}$ "positive definiteness"

Notice how much freedom we have to define this operation!

Once we have it, we obtain an inner product space.

Vector space with an inner product.

Examples:

① $\mathbb{R}^3$ $\vec{v} = (v_1, v_2, v_3)$ $(\vec{v}, \vec{w}) = v_1 w_1 + v_2 w_2 + v_3 w_3$

$\vec{w} = (w_1, w_2, w_3)$

- conjugate property ✓ (trivial)
- linearity ✓ (easy to see)
- positivity $\longrightarrow (\vec{v}, \vec{v}) = v_1^2 + v_2^2 + v_3^2$ ✓

② Space $\mathbb{V}$ generated by real linear combinations of

$$X = \{1, x, x^2, \dots, x^N\} \longrightarrow \text{LI}$$

$\vec{v}_0$ $\vec{v}_1$ ...

spans $\mathbb{V}$ by def.

Every $\vec{v}$ in $\mathbb{V}$ is such that

$$\vec{v} = \sum_{i=0}^{N} c_i \vec{v}_i \quad , \quad c_i \in \mathbb{R}$$

$$\updownarrow$$

$$f(x) = \sum_{i=0}^{N} c_i x^i \quad x \in (-\infty, \infty)$$

In this space of polynomials, we can then define

$$(f, g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-x^2}\, dx ,$$

which always converges (as long as N is finite), and you can check satisfies all the properties of the scalar product definition.

- We could have defined this product differently!

E.g. $$(f, g) = \int_0^\infty f(x)\, g(x)\, e^{-x}\, dx$$

or $$(f, g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, \frac{1}{1 + x^{2(N+1)}}\, dx$$

- More generally, any weight function $W(x) > 0$ such that $\int_a^b f(x) g(x) W(x)\, dx$ converges, is allowed and yields a good inner product.

What's the use of having an inner product space?

- Orthogonality makes sense:

$\vec{v}, \vec{w}$ are orthogonal iff $(\vec{v}, \vec{w}) = 0$

- Magnitude of a vector:

$$|\vec{v}| \equiv \sqrt{(\vec{v}, \vec{v})}$$

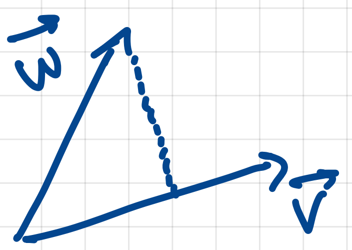
$\to \sqrt{v_1^2 + v_2^2 + v_3^2}$ in $\mathbb{R}^3$

$\to \left(\int_a^b f(x) f(x) W(x)\, dx \right)^{1/2}$

- Projection:

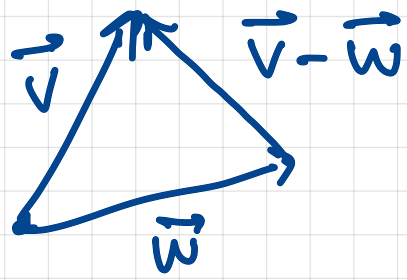
$$-1 \leq \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|\,|\vec{w}|} \leq 1$$

$\to$ "How much of $\vec{v}$ points in the direction of $\vec{w}$ (or viceversa)"



- **Distance:**

$$\left[(\vec{v}-\vec{w}, \vec{v}-\vec{w})\right]^{1/2} \geq 0 \qquad \text{distance between } \vec{v} \text{ and } \vec{w}.$$



- **Cauchy-Schwartz inequality** proof:

$$|(\vec{v}, \vec{w})| \leq |\vec{v}|\,|\vec{w}|$$

Use a smart linear combination:

① $\vec{v}\,|\vec{w}| - \vec{w}\,|\vec{v}| = \vec{x} \qquad (\vec{x}, \vec{x}) \geq 0$

Expanding,

$$2|\vec{v}|^2|\vec{w}|^2 - 2\,|\vec{v}||\vec{w}|(\vec{v}, \vec{w}) \geq 0$$

$$\cancel{2}|\vec{v}|^{\cancel{2}}|\vec{w}|^{\cancel{2}} - \cancel{2}\,\cancel{|\vec{v}|}\cancel{|\vec{w}|}(\vec{v}, \vec{w}) \geq 0$$

$$|\vec{v}|.|\vec{w}| - (\vec{v}, \vec{w}) \geq 0$$

② $\vec{v}\,|\vec{w}| + \vec{w}\,|\vec{v}| = \vec{y} \qquad (\vec{y}, \vec{y}) \geq 0$

$$|\vec{v}|.|\vec{w}| + (\vec{v}, \vec{w}) \geq 0$$

QED.

- **Triangle inequality**

$$|\vec{v}+\vec{w}| \leq |\vec{v}| + |\vec{w}|$$