Short Takes 331

Bases

Definition

A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N\}$ is said to be a basis of a vector space $\mathbb{V}$ if and only if:

1. The $\vec{v}_i$ are linearly independent
2. The $\vec{v}_i$ span $\mathbb{V}$

$\longrightarrow$ any vector in $\mathbb{V}$ can be written as a linear combination of the $\vec{v}_i$.

- Conceptually, a basis is a minimal set of vectors needed to generate a given vector space.

  - For example:

    $\vec{v}_1 = (1, 0)$; $\vec{v}_2 = (0, 1)$ together are a basis for $\mathbb{R}^2$

    Neither $\vec{v}_1$ nor $\vec{v}_2$ can, by themselves, generate $\mathbb{R}^2$.

  - Another example:

    $\vec{v}_1 = 1$; $\vec{v}_2 = 3x^2$; $\vec{v}_3 = \frac{1}{2}x$

    together generate a space of all polynomials of degree ≤ 2.

    If we add $\vec{v}_4 = 7x + 3$, we still generate the same space, but the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is not L.I.

    Indeed, $\vec{v}_4 = 14\vec{v}_3 + 3\vec{v}_1$.

    Therefore, we have at least one too many vectors!

* **One more example:**

$\vec{v}_1 = (2, 0, 0)$; $\vec{v}_2 = (1, 1, 1)$ $\mathbb{R}^3$

- Are they LI? Yes!

$$a\vec{v}_1 + b\vec{v}_2 = (2a + b, b, b) \overset{!}{=} 0$$

$$\Rightarrow b = 0, \quad a = 0 \quad ✓$$

- They are elements of $\mathbb{R}^3$. Do they span it?

No! For instance, it is not possible to generate a vector such as

$$(0, 1, 0)$$

Indeed! $a\vec{v}_1 + b\vec{v}_2 = (2a + b, b, b) \overset{!}{=} (0, 1, 0)$

$$\Rightarrow \begin{array}{l} b = 0 \\ b = 1 \end{array} \quad ✗$$

* **Last one:**

$\vec{v}_1 = \sin x$, $\vec{v}_2 = \sin 2x$, $\vec{v}_3 = \sin 3x$, ...

These are LI

What space do they span? More on this later!

Note: In a given vector space there will be an infinite number of choices for basis vectors.

For example, in $\mathbb{R}^2$, we could use

$\{(1,0), (0,1)\}$, or $\{(1,1), (1,0)\}$, or

$\{(-1,2), (\pi, 1)\}$, and so on

However, the number of vectors in a basis is always the same and depends only on the vector space in question...

Dimension: The dimension of a vector space $\mathbb{V}$ is given by the number of vectors in a basis.

We won't show it here, but this definition makes sense only if we can prove that all bases in a given space $\mathbb{V}$ indeed share the same number of vectors.

- We will make the following plausibility comments:
  - If a basis has too few vectors, it won't span the full space $\mathbb{V}$ and so it can't be a basis.
  - If a basis has too many vectors, it must be that at least one of them can be written in terms of the others as a linear combination, i.e. the set is linearly dependent and so it can't be a basis.

The proof that all bases have the same number of vectors then proceeds by assuming there are two good bases with different numbers of elements and showing that one arrives at a contradiction (e.g. we find that they can't both be LI).

- Finally, what happened to orthogonality?