Short Takes 331

Linear independence

Definition

Take N vectors in your vector space $\mathbb{V}$
$A_N = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N\}$. Is it possible to make the combination

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_N\vec{v}_N$$

vanish with non-zero coefficients a_j?

If not, then the set A_N is linearly independent.
If yes $\rightarrow$ linearly dependent

Note: it is always possible to set

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_N\vec{v}_N = 0 \quad !$$

Just take all $a_j = 0$! The issue is whether we can do that without setting all the a_j coefficients to zero.

Example

$$\vec{V}_1 = (1, 0) \qquad \vec{V}_2 = (0, -2)$$

$$a_1\vec{V}_1 + a_2\vec{V}_2 = (a_1, -2a_2)$$

Can we set this to $(0, 0)$?

Yes, but only if $a_1 = 0$ & $a_2 = 0$.

$\longrightarrow$ $\{\vec{V}_1, \vec{V}_2\}$ is a linearly independent set.
(or "$\vec{V}_1, \vec{V}_2$ are linearly independent")

Example

$\vec{v}_1 = (1, 0)$ $\qquad$ $\vec{v}_2 = (-1, 0)$

$$a_1\vec{v}_1 + a_2\vec{v}_2 = (a_1 - a_2, 0) \overset{!}{=} (0, 0)$$

$$\Rightarrow a_1 - a_2 = 0 \Rightarrow a_1 = a_2$$

So any value of a_1 will do the job (not only $a_1 = 0$), as long as we set $a_2 = a_1$.

Therefore, $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent.

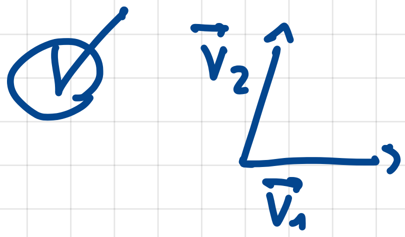
Example Are x & x^2 LI?

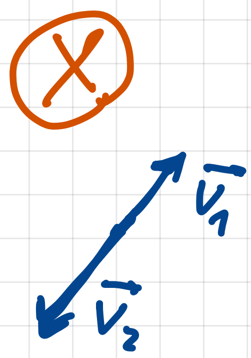
Yes!

The only way to make $ax + bx^2 = 0$ (for all values of x) is to set $a = b = 0$.

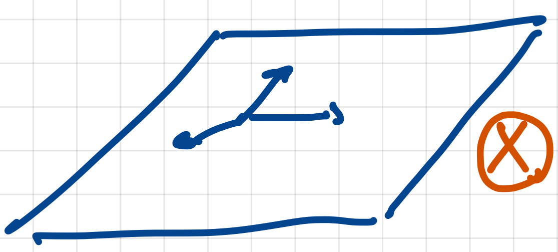
What does it mean?

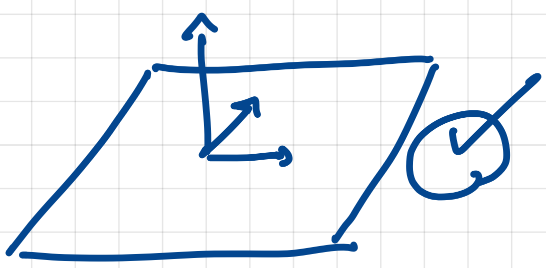
- Geometrically... two vectors are LI when they are not collinear





In the case of three vectors, LI means that they are not all on the same plane





- Algebraically ... LI means that it is not possible to write one of the vectors as a linear combination of the others.

Indeed... Suppose $a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_N \vec{v}_N = 0$

and $a_3 \neq 0$ (say, i.e. the set is LD)

then $\vec{v}_3 = \frac{-1}{a_3}\left(a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_4 \vec{v}_4 + \dots + a_N \vec{v}_N\right)$

$\rightarrow$ This is not possible to do if all $a_j = 0$ (LI).

Notes:

- In the case of two vectors, LD means one of them is proportional to the other: $\vec{v}_2 = \text{const.}\,\vec{v}_1$.
- Any set containing $\vec{0}$ is LD, since we always have

$$a\,\vec{0} = \vec{0} \qquad \text{for } \underline{\text{any}} \text{ constant } a.$$

- In general, we call a linear combination the expression

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_N \vec{v}_N$$

$\hookrightarrow$ This is a linear combination of the $\vec{v}_j$ vectors.

Another way to write it:

$$\sum_{j=1}^{N} a_j \vec{v}_j$$

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