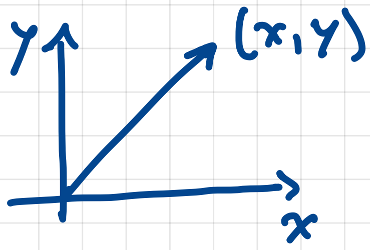
Short Takes 331

Vector spaces: Examples

Simple examples: $\mathbb{R}^2$



$, x, y \in \mathbb{R}$

with $\mathbb{R}$ as scalars

vector addition: $(x,y) + (x',y') = (x+x', y+y')$

vector-scalar multiplication: $q \cdot (x,y) = (qx, qy)$

- $\mathbb{C}^2$: (z,w) , $z, w \in \mathbb{C}$

  with $\mathbb{C}$ or $\mathbb{R}$ as scalars

- $\mathbb{R}^m$ $\mathbb{C}^n$, with appropriate scalars.

- More generally...

  $\mathbb{R}^{n \times m}$, for example $\mathbb{R}^{2 \times 2}$

  "matrices" $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$

How is '+' defined?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ t & r \end{pmatrix} = \begin{pmatrix} a+x & y+b \\ c+t & d+r \end{pmatrix}$$

What about scalar multiplication?

$q \in \mathcal{S}$

$$q \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} qa & qb \\ qc & qd \end{pmatrix}$$

... and similarly $\mathbb{C}^{n \times m}$

More interesting examples: spaces of functions

- Polynomials

Example: {All polynomials of degree ≤ 3 (or any fixed number)}

$$V = \{ a + bx + cx^2 + dx^3 \ ; \ \underbrace{a, b, c, d}_{} \in \mathbb{R} \}$$

with $\mathbb{R}$ as scalars.

these determine a given vector

+ vector addition is simply polynomial addition (order by order):

$$\begin{array}{r} 1 + x + 3x^2 \\ + \quad 5x^2 - x^3 \\ \hline 1 + x + 8x^2 - x^3 \end{array} \leftarrow \text{also in } V$$

+ scalar-vector multiplication is also simple:

$$2 \cdot (5x^2 - x^3) = 10x^2 - 2x^3$$

+ Not allowed: multiplication of two polynomials

$$(1+x) \cdot x = \underbrace{x + x^2}_{}$$

$\longrightarrow$ would be in V, but recall vec-vec multiplication is not part of our definition.

- More general functions

  - Example: $\mathbb{V} = \{$All functions of the form
  $$f(x) = a\cos x + b\cos 2x + c\sin x, \quad a, b, c \in \mathbb{R}\}$$
  with $\mathbb{R}$ as scalars.

    - Here, as with polynomials, we define addition of vectors in the same way as addition of functions $f(x) + g(x)$.
    - Similarly for multiplication by a scalar.

- Generalizations

  - You can use any functions, e.g. e^{-x}, $\tan x$, etc. and in fact you can use an infinite number of them!
  - You can generalize this to multivariable functions too! $f(x,y)$ or $f(x,y,z)$; e.g.
  $\mathbb{V} = \{$All functions of the form
  $$f(x,y) = a\,xy + b\,x^2 + c\,y^2x, \quad a, b, c \in \mathbb{C}\}$$

- In QM:
$\mathbb{V} = \{$ All functions $f(x)$, $x \in \mathbb{R}$, such that
$$\int_{-\infty}^{\infty} |f(x)|^2\, dx \text{ is finite} \}$$