Short Takes 331

Fourier series: first concepts

Fourier series : a conceptual introduction

- You know about Taylor series

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

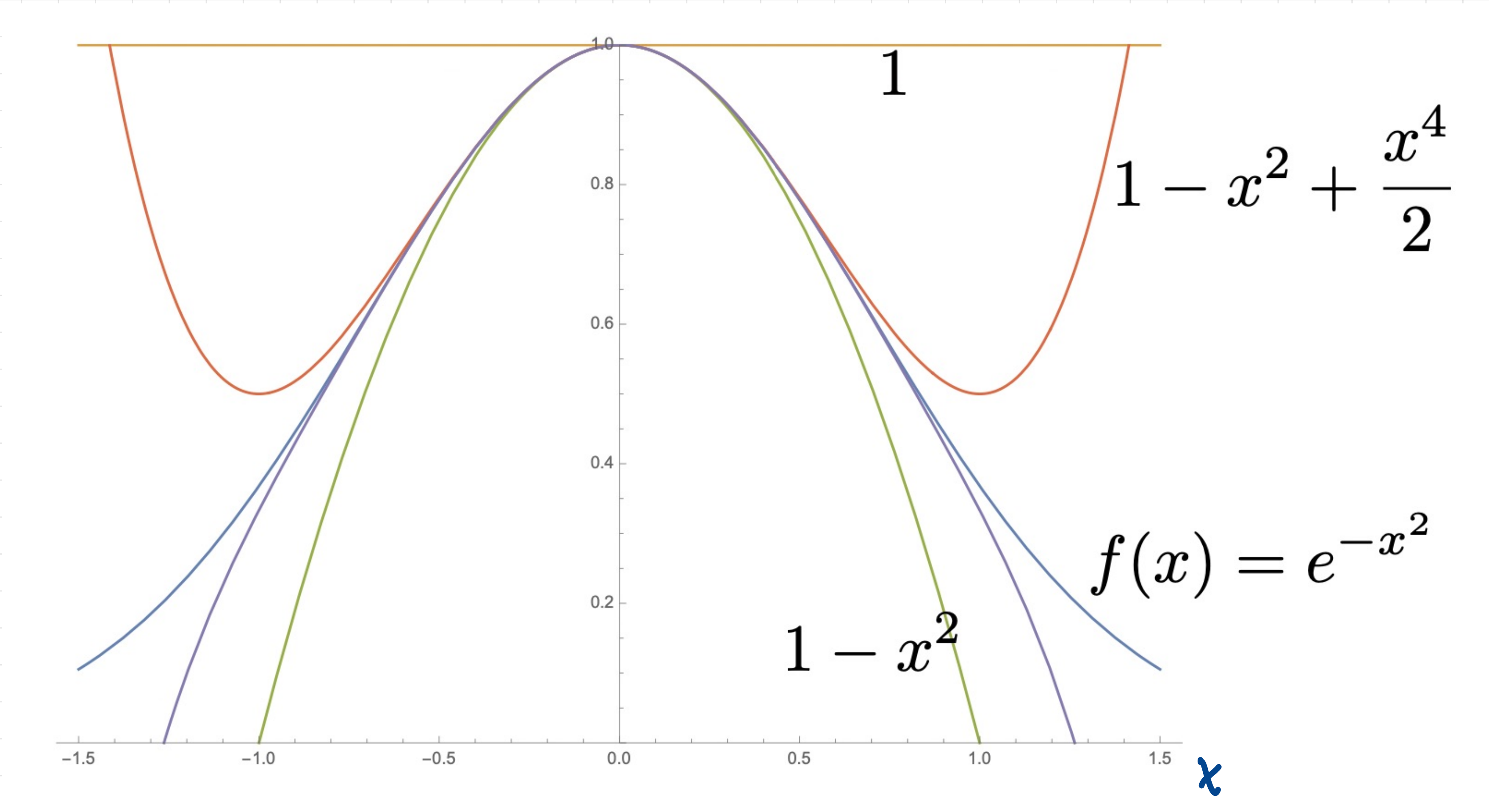
$\rightarrow (a_0, a_1, a_2, \dots)$ are the coordinates of $f(x)$:

in the basis $X = \{1, (x-x_0), (x-x_0)^2, \dots$

Different x_0 ref points give diff coefficients

$$\begin{cases} a_0 = f(x_0) \\ a_1 = \dfrac{df}{dx}(x_0) \\ a_2 = \dfrac{1}{2!}\dfrac{d^2 f}{dx^2}(x_0) \\ \vdots \end{cases}$$

This is a local expansion around a specific point.



$x_0 = 0$

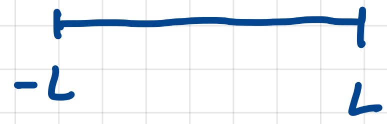
$f(x_0) = 1 \longrightarrow (a_0, a_1, a_2, \dots) = (1, 0, -1, 0, \dots)$

$f'(x_0) = 0$

$f''(x_0) = -2$

$\vdots$

- The Fourier series is just another basis



$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{k\pi x}{L}\right) + B_k \sin\left(\frac{k\pi}{L} x\right) \right]$$

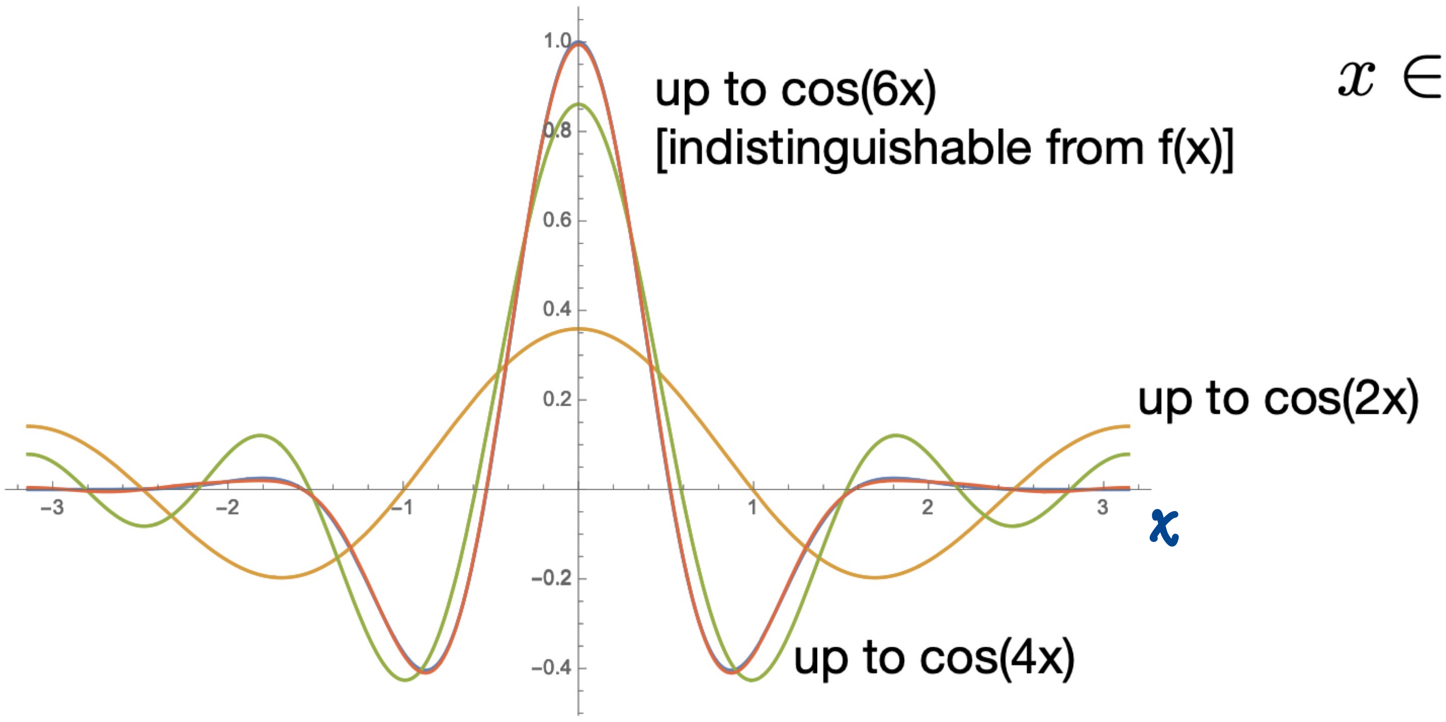
Here the coordinates are A_k, B_k (note there's no B_0)
and the basis is

$$X_F = \left\{ 1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi}{L} x\right), \cos\left(\frac{2\pi}{L} x\right), \sin\left(\frac{2\pi}{L} x\right), \dots \right\}$$

And the coefficients are calculated via ...

$$\begin{cases} A_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx \\ A_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi}{L} x\right) dx , \quad B_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi}{L} x\right) dx \end{cases}$$

$$f(x) = e^{-x^2} \cos(3x)$$
$$= 0.02973 + 0.10893 \cos(x) + 0.22024 \cos(2x) + \dots$$



$x \in [-\pi, \pi]$

$L = \pi$

$f(x) = e^{-x^2} \cos(3x)$ is an even function $\Rightarrow B_k = 0 \;\; \forall k$.

$\frac{A_0}{2} = 0.02973\dots$

$A_1 = 0.10893\dots$

$A_2 = 0.22024\dots$

- Why use the Fourier basis? Looks complicated!

→ Convenience

→ physical reasons

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$$

$e^{\pm ikx}$ ← "plane wave"
eigenfunction of $-\frac{d^2}{dx^2}$

More specifically, our Fourier basis

$$X_F = \left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi}{L}x\right), \cos\left(\frac{2\pi}{L}x\right), \sin\left(\frac{2\pi}{L}x\right), \dots\right\}$$

are eigenfunctions of
$-\frac{d^2}{dx^2}$ with $x \in [-L, L]$
and periodic boundary conditions
$f(-L) = f(L)$