Short Takes 331

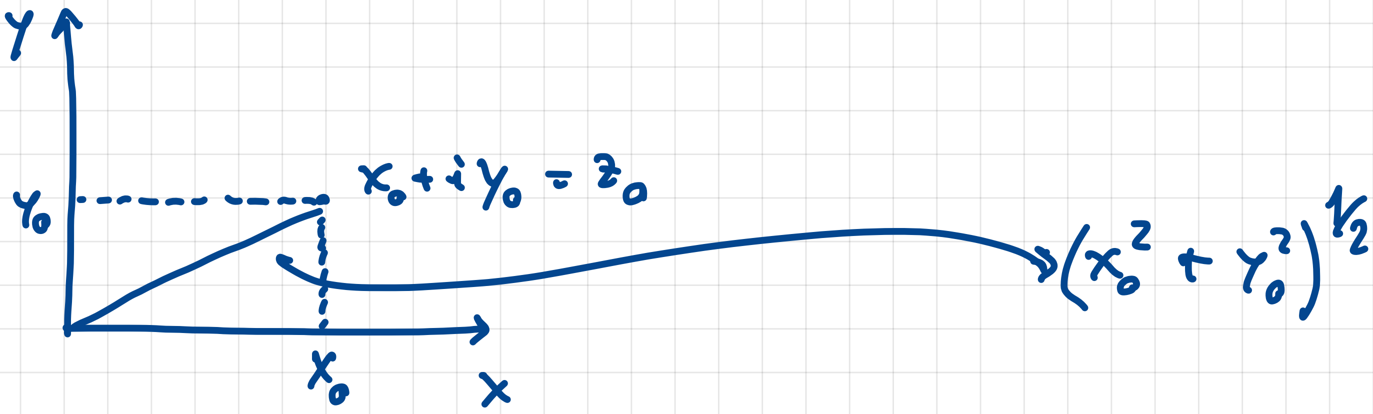
Complex numbers basics

Complex numbers : basics

$z = x + iy \quad , \quad x, y \in \mathbb{R} \quad , \quad i^2 = -1$

$x = \mathrm{Re}(z)$ "Real part of z"

$y = \mathrm{Im}(z)$ "Imaginary part of z"



- **Basic operations** $\quad z_1 = x_1 + i y_1 \quad z_2 = x_2 + i y_2$

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$$

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}$$

$$z_2^* = x_2 - i y_2$$

Complex conjugation

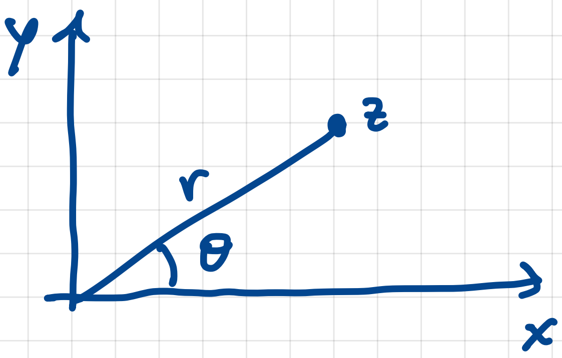
$$z^* = x - iy \quad , \quad \text{if} \quad z = x + iy$$

$$z^* z = (x - iy)(x + iy) = x^2 - (iy)^2 = x^2 + y^2$$

- **Polar representation**

$$z = r\cos\theta + i r \sin\theta = r e^{i\theta}$$

Euler's formula: $e^{i\phi} = \cos\phi + i \sin\phi$



$$r = (x^2 + y^2)^{1/2}$$

$$\theta = \arctan(y/x)$$

- **Example**: $z^n = r^n (\cos\theta + i\sin\theta)^n = r^n e^{in\theta}$

$$= r^n (\cos n\theta + i \sin n\theta)$$

$$\sum_{k=0}^{n} \binom{n}{k} \cos^k\theta \, (i\sin\theta)^{n-k}$$

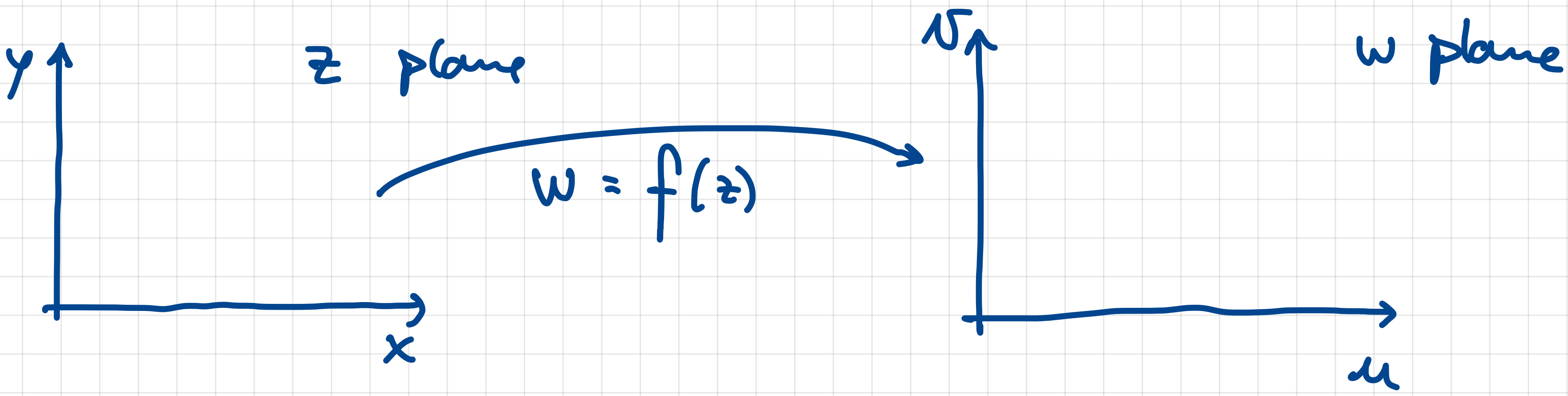
(binomial theorem)

Formulas for $\cos n\theta$ and $\sin n\theta$.

$n=2$:

$$\begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\cos\theta \sin\theta \end{cases}$$

- **Functions of a complex variable**



$$f(z) = u(x,y) + i\,v(x,y)$$

- Can we differentiate w.r.t. z?
- How do we do integrals?
- What about Taylor expansions?

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = ?$$

$\rightarrow$ We can take $\Delta z = \Delta x$ real

or $\Delta z = i\Delta y$ imaginary

$\rightarrow$ if the limit exists, we must obtain the same result

$$\rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

"Cauchy-Riemann equations"

(necessary but not sufficient!)

- Generally speaking, derivatives work as you would expect:

$$\frac{d}{dz}(z^n) = n z^{n-1} \qquad \frac{d(\sin z)}{dz} = \cos z \qquad \frac{d(\cos z)}{dz} = -\sin z$$

$$\frac{d}{dz}(e^z) = e^z$$

... as do series expansions in these well behaved cases.

! Be careful with $\log z$, fractional powers of z.

- Example

$$f(z) = z^2 + 1 \qquad \frac{\partial f}{\partial z} = 2z$$

$$= \underbrace{x^2 - y^2 + 1}_{u(x,y)} + i\, \underbrace{2xy}_{v(x,y)} \longrightarrow \begin{aligned} \frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x} \end{aligned} \quad ✓$$