Short Takes 331

More about Gaussian integrals

More about Gaussian integrals : a step towards QFT & stat mech.

- In a previous video...

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda x^2} dx = \sqrt{\frac{2\pi}{\lambda}}$$

- Also, $\displaystyle\int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_d \ \exp\left(-\frac{1}{2} x^T M x\right) = \left(\frac{(2\pi)^d}{\det M}\right)^{1/2}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \qquad x^T = (x_1 \ x_2 \ \dots \ x_d)$$

This works if the real part of all the eigenvalues of M is positive.

- Note

$$x^T M x = \sum_{ij} x_i M_{ij} x_j = \sum_{ij} x_i (M_S)_{ij} x_j$$

$$M = \underbrace{\frac{M+M^T}{2}}_{M_S} + \underbrace{\frac{M-M^T}{2}}_{M_A}$$

$$\sum_{ij} (M_A)_{ij} x_i x_j = 0$$

$M_S = M_S^T$: symmetric part

$M_A = -M_A^T$: anti-symmetric part

→ We only care about the symmetric part of M.

(the anti-symmetric part does not contribute to the integral)

From now on, we can assume M is symmetric.

• A more general case ...

$$I(y) = \int_{-\infty}^{\infty} dx_1 \, dx_2 \dots dx_d \; e^{-\left(\frac{1}{2} x^T M x \, - \, x^T y\right)}$$

$$\begin{aligned}
x^T M x - x^T y &= x^T M x - x^T \underline{M M^{-1}} y \\
&= x^T M (x - M^{-1} y) \\
&= (x - \underline{M^{-1} y})^T M (x - M^{-1} y) + \underline{(M^{-1} y)^T M (x - M^{-1} y)} \\
&= \bar{x}^T M \bar{x} + (M^{-1} y)^T M x - y^T (M^{-1})^T \underline{M M^{-1}} y \\
&= \bar{x}^T M \bar{x} + y^T (M^{-1})^T M x - y^T M^{-1} y \\
&= \bar{x}^T M \bar{x} + y^T x - y^T M^{-1} y
\end{aligned}$$

$\bar{x} = x - M^{-1} y$

$\bar{x}_i = x_i - (M^{-1} y)_i$

$d\bar{x}_i = dx_i$

$$\Rightarrow \quad x^T M x - 2 x^T y = \bar{x}^T M \bar{x} - y^T M^{-1} y$$

Therefore,

$$I(y) = \int_{-\infty}^{\infty} d\bar{x}_1 \, d\bar{x}_2 \dots d\bar{x}_d \; e^{-\frac{1}{2} \bar{x}^T M \bar{x}} \; e^{\frac{1}{2} y^T M^{-1} y}$$

$$= \left(\frac{(2\pi)^d}{\det M} \right)^{1/2} e^{\frac{1}{2} y^T M^{-1} y}$$

or,

$$\boxed{I(y) = I(0) \, e^{\frac{1}{2} y^T M^{-1} y}}$$

Generating functional of moments of the Gaussian distribution.

$$\left. \frac{\partial^m I(y)}{\partial y_{i_1} \dots \partial y_{i_m}} \right|_{y=0} = \int_{-\infty}^{\infty} dx_1 \dots dx_d \; x_{i_1} \dots x_{i_m} \; e^{-\frac{1}{2} x^T M x}$$

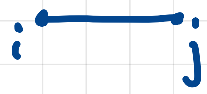
$$= I(0) \left(M^{-1}_{x_{i_1} x_{i_2}} \dots \right)$$

m must be even.

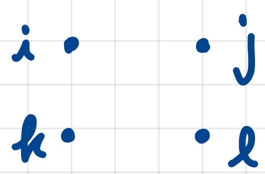
• Quick example

$$\left.\frac{\partial^2 I(y)}{\partial y_i \, \partial y_j}\right|_{y=0} = G_{ij} = \int dx_1 dx_2 \dots dx_d \; x_i \, x_j \; e^{-\frac{1}{2} x^T M x}$$

$$= I(0) \, . \, M^{-1}_{ij}$$



$$\left.\frac{\partial^4 I(y)}{\partial y_i \, \partial y_j \, \partial y_k \, \partial y_l}\right|_{y=0} = I(0)\left(M^{-1}_{ij} M^{-1}_{kl} + M^{-1}_{ik} M^{-1}_{jl} + M^{-1}_{il} M^{-1}_{jk}\right)$$



• Why everything in terms of M^{-1} ?

$$\ln\left(\frac{I(y)}{I(0)}\right) = \frac{1}{2} y^T M^{-1} y$$

Generating function of cumulants.

• Usually, in QFT and stat mech,

$$\mathcal{Z}[j] = \int \mathcal{D}x \; P[x] \exp\left[\sum_i j_i \, x_i\right]$$

Sources

$$W[j] = \ln\left(\frac{\mathcal{Z}[j]}{\mathcal{Z}[0]}\right)$$