Short Takes 331

The adjoint operator

The adjoint operator (aka Hermitian adjoint)

- With matrices (i.e. linear operators in a finite-dimensional space)

$$M^\dagger: \quad [M^\dagger]_{ij} = M^*_{ji}$$

- More generally, the adjoint operator is defined using an inner product:

$$(\vec{v}, \vec{w}) = s \quad , \quad s \in \mathbb{S} \text{ field of scalars}$$

$\vec{v}, \vec{w} \in \mathbb{V}$ are vectors

... such that:

1- $(\vec{v}, \vec{w}) = (\vec{w}, \vec{v})^*$ "Conjugate property"

2- $(x\vec{v}, \vec{w}) = x(\vec{v}, \vec{w})$,
$x \in \mathbb{S}$ a scalar
$(\vec{u} + \vec{v}, \vec{w}) = (\vec{u}, \vec{w}) + (\vec{v}, \vec{w})$ "linearity"

3- $(\vec{v}, \vec{v}) \geq 0$
if $(\vec{v}, \vec{v}) = 0$ then $\vec{v} \equiv \vec{0}$ "positive definiteness"

- The adjoint $M^\dagger$ is then defined via...

$$(\vec{v}, M^\dagger \vec{w}) = (M\vec{v}, \vec{w})$$

- In conventional, finite-dim spaces, with $(\vec{v}, \vec{w}) = \sum_k v_k^* w_k$,

$$(M\vec{v}, \vec{w}) = \sum_{i,j} M^*_{ij} v_j^* w_i = \sum_{i,j} v_j^* M^\dagger_{ji} w_i ,$$

therefore $M^{+}_{ij} = M^{*}_{ji}$.

- In cases involving diff. ops, it may be more complicated...

$$(f, Dg) = \int_{-\infty}^{\infty} f^{*}(x)\, \frac{dg}{dx}(x)\, dx = -\int_{-\infty}^{\infty} \frac{df^{*}}{dx}(x)\, g(x)\, dx = (-Df, g)$$

$D = \frac{d}{dx}$ (e.g.)

int x parts

So.. $D^{+} = -\frac{d}{dx} \neq D$!

In QM, e.g. $\hat{p} = \frac{\hbar}{i}\frac{d}{dx} = \hat{p}^{+}$

⊛ Note: formally, the definition of the adjoint in infinite dimensional spaces must be handled with some care.
I'm leaving out lots of details just to get some practical information across.

- The adjoint inherits properties from the inner product...

  - $(A^{+})^{+} = A$
  - $(A^{+})^{-1} = (A^{-1})^{+}$
  - $(A + B)^{+} = A^{+} + B^{+}$
  - $(AB)^{+} = B^{+}A^{+}$

E.g.

$$(\vec{v}, (A^{+})^{+}\vec{w}) = (A^{+}\vec{v}, \vec{w}) = (\vec{w}, A^{+}\vec{v})^{*} = (A\vec{w}, \vec{v})^{*} = (\vec{v}, A\vec{w})$$

$$\Rightarrow (A^{+})^{+} = A$$