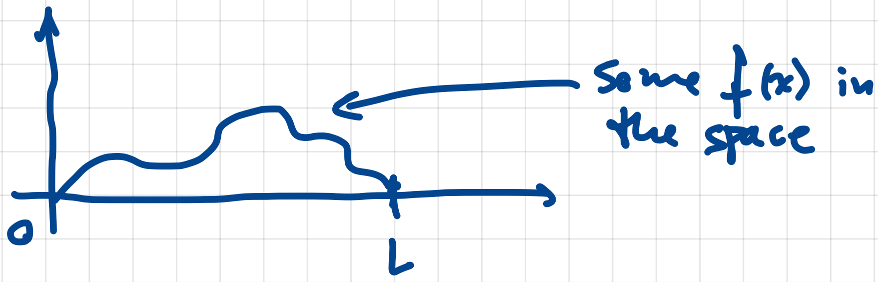
Short Takes 331

A 1d Green's function example two ways

A 1D Green's function example two ways

Consider the op $\hat{A} = -\frac{d^2}{dx^2}$, $f(0) = f(L) = 0$



- **Green's function?**

$$\boxed{-\frac{d^2 G(x,x')}{dx^2} = \delta(x-x')}$$

- Why is this important/useful?

  Armed with this result, we can solve $-\frac{d^2 f}{dx^2} = h(x)$ via "convolution":

$$f(x) = \int_0^L G(x,x')\, h(x')\, dx' , \qquad \text{for any } h(x)$$

- something weird at $x = x'$!

- Think of x' as a parameter and x as the variable.

- $$\int_{x'-\varepsilon}^{x'+\varepsilon} -\frac{d^2 G(x,x')}{dx^2}\, dx = -\left[\frac{dG(x'+\varepsilon, x')}{dx} - \frac{dG(x'-\varepsilon, x')}{dx}\right] =$$

$$= \int_{x'-\varepsilon}^{x'+\varepsilon} \delta(x-x')\, dx = 1 \longrightarrow$$ G' is discontinuous at $x = x'$

- We will assume G itself has no problems at $x = x'$, i.e. G is continuous at $x = x'$.

How do we find $G(x,x')$?

- Region $x<x'$: $-\dfrac{d^2G(x,x')}{dx^2} = 0 \Rightarrow G(x,x') = A(x').x + B(x')$

  i.e. it's linear in x !

  Moreover, $G(0,x') = 0 \Rightarrow B \equiv 0$

  $$\boxed{G(x,x') = A(x')\,x}$$

- Region $x>x'$: Similar thing $\rightarrow G(x,x') = C(x').x + D(x')$

  And $G(L,x') = 0 \Rightarrow C(x')L + D(x') = 0$

  $\Rightarrow D(x') = -C(x')L$

  $$\boxed{G(x,x') = C(x')(x-L)}$$

- At $x=x'$:
  - continuity : $A(x)x = C(x)(x-L) \Rightarrow A(x) = C(x)\left(\dfrac{x-L}{x}\right)$
  - discontinuity in G' : $C(x) - A(x) = -1 \Rightarrow A(x) = C(x)+1$

Therefore, $C(x)+1 = C(x)\dfrac{(x-L)}{x} \Rightarrow C(x) = \dfrac{-x}{L}$

Finally,

$$\boxed{G(x,x') = \begin{cases} \left(1-\dfrac{x'}{L}\right).x & x<x' \\ \left(1-\dfrac{x}{L}\right).x' & x \geqslant x' \end{cases}}$$

$\rightarrow$ positive slope in x

$\rightarrow$ negative slope in x

Note: $G(x,x') = G(x',x)$

- **Another way to find G.**

Recall the problem...

$$\hat{A} = -\frac{d^2}{dx^2} \quad , \quad f(0) = f(L) = 0$$

0 —— L

We are looking for the inverse operator.

We know the **eigenvectors & eigenvalues**!

$$\psi_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L} x\right) \quad , \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2 \quad , \quad k = 1, 2, \ldots$$

Then

$$\hat{A}^{-1} h(x) = \int_0^L G(x, x') \, h(x') \, dx'$$

$$\boxed{G(x,x')} = \sum_{k=1}^{\infty} \psi_k(x) \, \lambda_k^{-1} \, \psi_k^*(x')$$

$$= \boxed{\frac{2}{L} \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L} x\right) \left(\frac{L}{k\pi}\right)^2 \sin\left(\frac{k\pi}{L} x'\right)} \quad !$$

$L = 1$

$x' = 1/3$