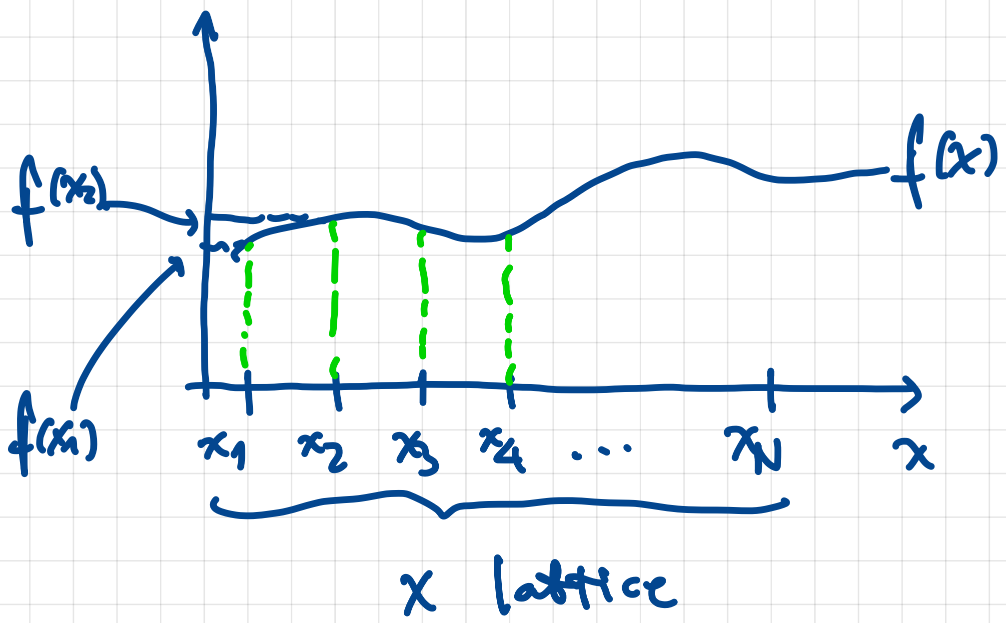
Short Takes 331

Diagonalizing differential operators on the lattice

Diagonalizing differential operators on the lattice

- How do we represent $f(x)$ on a computer?



$$f(x) \longrightarrow \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{pmatrix} = \vec{f}$$

or simply $f_1, \dots, f_N$

- How do we represent derivatives $f'(x), f''(x)$, etc?

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \xleftrightarrow{\text{on the lattice}} \quad \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

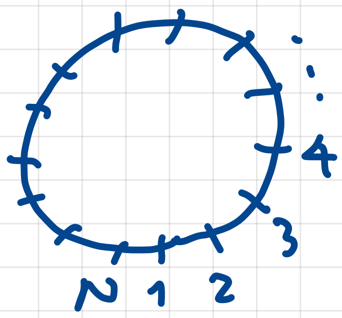
"forward derivative"

or, $\dfrac{f_{i+1} - f_i}{\ell}$, assuming uniform lattice where $x_{i+1} - x_i = \ell$ ← lattice spacing.

- As a linear operator, ie a matrix...

$$\frac{df}{dx} \longrightarrow \frac{1}{\ell} \underbrace{\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & & \ddots & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix}}_{N \times N} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$$

rep of $d/dx \longrightarrow D^{(1)}$



boundary condition! (periodic in this case) → Usually, when solving diff eq, we need to think about these separately. Here, it comes with the matrix!

Notice...

- $D^{(1)} \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, for <u>any</u> c
  $\rightarrow$ $\ker D^{(1)}$ is non-trivial, so we don't have a well-defined inverse op.

  Makes sense!
  $\frac{dc}{dx} = 0 \quad \forall c$

- $\left[D^{(1)}\right]_{ij} = \frac{1}{\ell}\left(-\delta_{ij} + \delta_{i+1,j}\right)$ (with periodic indexing $N+1 \leftrightarrow 1$)

- $\underbrace{D^{(1)}\vec{v}_k}_{\downarrow} = ?$, $\left[\vec{v}_k\right]_j = \frac{1}{\sqrt{N}}\exp\left(i\frac{2\pi}{N}k.j\right)$

$$[\quad]_q = \sum_{j=1}^{N} \frac{1}{\ell}\left(-\delta_{qj} + \delta_{q+1,j}\right)\cdot\frac{1}{\sqrt{N}}\exp\left(i\frac{2\pi}{N}k\cdot j\right)$$

$$= \frac{1}{\ell}\frac{1}{\sqrt{N}}\left[-\exp\left(i\frac{2\pi}{N}k\cdot q\right) + \exp\left(i\frac{2\pi}{N}k.(q+1)\right)\right]$$

$$= \frac{1}{\ell}\left[\exp\left(i\frac{2\pi}{N}k\right) - 1\right]\frac{1}{\sqrt{N}}\exp\left(i\frac{2\pi}{N}kq\right)$$

$$= \lambda_k\cdot\left[\vec{v}_k\right]_q$$

$\rightarrow$ $\boxed{\vec{v}_k \text{ is an eigenvector of } D^{(1)} \text{ with eigenvalue } \lambda_k = \frac{1}{\ell}\left[\exp\left(i\frac{2\pi}{N}k\right) - 1\right]}$

"Fourier modes are eigenvectors of derivative ops."
(check boundaries!)

- Note $\lambda_N = 0$ & $v_N = \frac{1}{\sqrt{N}}$ $\rightarrow$ constant!
(makes sense)

- Finally,

$$\vec{V}_{k'}^{\dagger} D^{(1)} \vec{V}_k = \lambda_k \delta_{kk'} \quad \text{because } \vec{V}_{k'}^{\dagger} \vec{V}_k = \delta_{kk'}$$

Therefore

$$U = \begin{pmatrix} \vec{V}_1 & \vec{V}_2 & \cdots & \vec{V}_N \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{pmatrix}$$

is the unitary matrix that diagonalizes $D^{(1)}$

$$U^{\dagger} D^{(1)} U = D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix}$$

$$U_{kj} = \frac{1}{\sqrt{N}} e^{i \frac{2\pi}{N} k \cdot j}$$

our old Fourier matrix!