Short Takes 331

A very special unitary matrix Part 2

A very special unitary matrix. Part 2.

$$\boxed{\begin{aligned}[F]_{jk} &= \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}jk} \\ &= \frac{1}{\sqrt{N}} w^{jk}\end{aligned}} , \quad w = e^{i2\pi/N}$$

$N \times N$ $\quad j,k = 1, 2, \ldots, N$

F: "Discrete Fourier transform matrix"

- Let's show it's indeed unitary!

$$[F^\dagger]_{kl} = [F]^*_{lk} = \frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}kl}$$

Then,

$$\sum_{k=1}^{N} F_{jk} F^\dagger{}_{kl} = \frac{1}{N} \cdot \sum_{k=1}^{N} e^{i\frac{2\pi}{N}(jk-kl)}$$

$$= \frac{1}{N} \sum_{k=1}^{N} e^{i\frac{2\pi}{N}k(j-l)}$$

- If $j = l \longrightarrow \frac{1}{N} \cdot \sum_{k=1}^{N} 1 = 1$
- If $j \neq l \longrightarrow \frac{1}{N} \sum_{k=1}^{N} \alpha^k = \frac{1}{N}\left(\frac{1-\alpha^{N+1}}{1-\alpha} - 1\right) = ⊛$

$$\alpha = e^{i\frac{2\pi}{N}(j-l)}$$

$$\frac{1-\alpha^{N+1}}{1-\alpha} - 1 = \frac{\cancel{1} - \alpha^{N+1} + \alpha \cancel{-1}}{1-\alpha} = \frac{\alpha(1-\alpha^N)}{1-\alpha} \longrightarrow 0$$

but...

$$\alpha^N = e^{i2\pi(j-l)} = 1$$

$$e^{i2\pi m} = 1 \quad \forall m \in \mathbb{Z}.$$

$\Rightarrow$ ⊛ $\rightarrow 0$

This shows that indeed, $\boxed{FF^\dagger = \mathbb{1}}$.

In the same way you can show $F^\dagger F = \mathbb{1}$.

$\Rightarrow$ F is unitary!

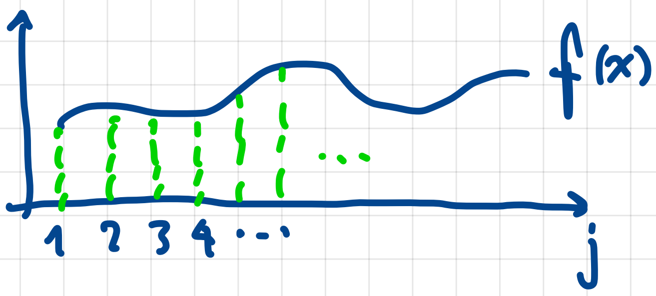
- How is F a <u>discrete Fourier transform</u>?

If $\vec{v}$ is a vector in our N-dimensional space,

$F\vec{v}$ is given by

$$[F\vec{v}]_k = \sum_{j=1}^{N} \frac{1}{\sqrt{N}} \exp\left(i \frac{2\pi}{N} k j\right) [\vec{v}]_j$$

- $\vec{v}$ may represent the N-point <u>discretization</u> of a function



$$[\vec{v}]_j = f(x_j) \equiv f_j$$

$$\tilde{f}_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i \frac{2\pi}{N} k j} f_j$$

Naively $O(N^2)$ operations, but with FFT $\rightarrow O(N \log N)$

- Note: in the previous video...

$$\underbrace{F^\dagger M F}_{} = D \quad \rightarrow O(N^3) \text{ vs } O(N^2 \log N)$$

regard as FT N columns,
each costing $O(N \log N)$ ops.