Short Takes 331

A very special unitary matrix

A very special unitary matrix: the discrete Fourier transform

- Unitary matrices: $U^{-1} = U^{\dagger} = (U^T)^*$

$$[U^{-1}]_{ij} = U^*_{ji} \qquad N \times N \qquad i,j = 1,2,\dots,N$$

- Moreover columns and rows are mutually orthonormal:

$$[\vec{V}_k]_i \equiv U_{ik} \qquad (k\text{-th column of } U) \quad , \quad k = 1,2,\dots,N$$

$$U = \begin{pmatrix} \vec{V}_1 & \vec{V}_2 & \dots & \vec{V}_N \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{pmatrix}$$

but we know

$$\delta_{ab} = \sum_i [U^{-1}]_{ai} U_{ib} = \sum_i U^*_{ia} U_{ib} = \sum_i [\vec{V}_a]^*_i [\vec{V}_b]_i$$

$$= (\vec{V}_a, \vec{V}_b) \checkmark$$

inner product!

- On the other hand, we also have completeness:

$$\delta_{ab} = \sum_k U_{ak} [U^{-1}]_{kb} = \sum_k [\vec{V}_k]_a [\vec{V}_k]^*_b$$

$\longrightarrow$ not really an inner product between two vectors!

unless... think of it as rows!

- $\det U = e^{i\alpha}$, for some $\alpha \in \mathbb{R}$, i: imaginary unit , $i^2 = -1$

$$e^{i\alpha} = \cos\alpha + i \sin\alpha \qquad \text{(Euler's formula)}$$

$$|e^{i\alpha}|^2 = \cos^2\alpha + \sin^2\alpha = 1$$

α parametrizes a complex number of unit magnitude.

$\det(U^{-1}U) = \det \mathbb{1} = 1 = \det U^\dagger U = \det U^\dagger \det U =$

$= (\det U)^* \det U = |\det U|^2 \Rightarrow |\det U| = 1$

- Unitary matrices form a group under multiplication $U(N)$

  essential property: $U_1 U_2$ is unitary if U_1 & U_2 are.

  Indeed, $(U_1 U_2)^\dagger = U_2^\dagger U_1^\dagger = U_2^{-1} U_1^{-1} = (U_1 U_2)^{-1}$

  This group will typically be non-abelian (ie non-commutative)

  $U_1 U_2 \neq U_2 U_1$

All of these properties (and a few more) we get just by virtue of U being unitary!

- So what's our very special unitary matrix?

$$\boxed{\begin{aligned}[F]_{jk} &= \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}jk} \\ &= \frac{1}{\sqrt{N}} w^{jk}\end{aligned}}\qquad N\times N \qquad j,k = 1, 2, \dots, N, \quad w = e^{i2\pi/N}$$

This particular structure allows extremely efficient algorithms like the "Fast Fourier transform".

For a large class of important matrices M,

$$\underbrace{F^\dagger M F}_{} = D \leftarrow \text{diagonal matrix!}$$

discrete rep of differential operators like $\frac{\partial^2}{\partial x^2}$

Takes $O(N^3)$ operations in general

but only $O(N^2 \log N)$ for F (with the fast Fourier transform algorithm)