# Short Takes 331

# The Dirac delta

# The Dirac delta

Remember the Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

The Dirac delta is similar ...

$$\delta(x) = 0 \quad \text{if } x \neq 0$$

and

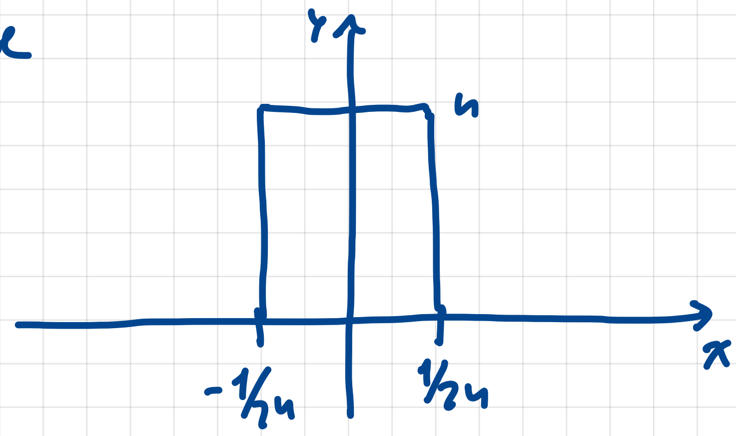
$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0) \quad ; \text{ in particular } \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

- $$\int_a^b f(x)\,\delta(x)\, dx = \begin{cases} f(0) & \text{if } a < 0 < b \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b f(x)\,\delta(x - x_0)\, dx = \begin{cases} f(x_0) & \text{if } a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

- How do we understand this?
  - Define it as a limit ...

$$d_n(x) = \begin{cases} n & \text{if } -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

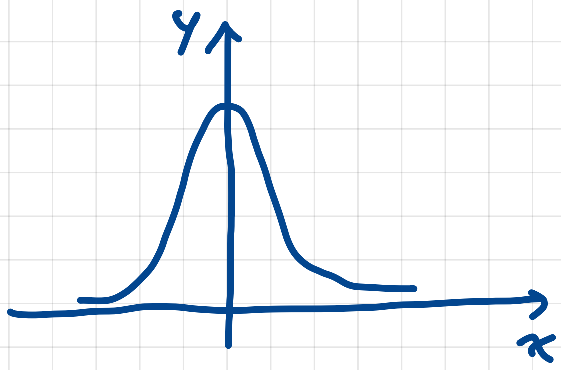



Then $\displaystyle\int_{-\infty}^{\infty} d_n(x)\, dx = n \cdot \left(\frac{1}{2n} - \left(-\frac{1}{2n}\right)\right) = n \cdot \frac{1}{n} = 1 \quad \forall n.$

As $n \to \infty$, $d_n(x) \to 0$ everywhere except at $x = 0$.

- Other reps:

  - <u>Gaussians</u>  $g_n(x) = A_n e^{-c_n x^2}$




    - make $c_n \to \infty$ as $n \to \infty$
    - set $A_n$ such that
      $$\int_{-\infty}^{\infty} g_n(x)\, dx = 1 \quad \text{for all } n.$$

  - <u>Lorentzians</u>
    $$l_n(x) = \frac{1}{\pi} \frac{b_n}{x^2 + b_n^2}$$
    - $\int_{-\infty}^{\infty} l_n(x)\, dx = 1$, so just make $b_n \to 0$ as $n \to \infty$.

- <u>Application</u>

  - $A\vec{v} = \vec{w}$ with matrices $\rightarrow$ $\vec{v} = A^{-1}\vec{w}$
    $AA^{-1} = A^{-1}A = \mathbb{1}$.  $\boxed{v_i = \sum_k (A^{-1})_{ik} w_k}$

  - $\hat{D}\phi(x) = f(x)$, $\hat{D}$ diff. op (e.g. $\nabla^2$)

    then

    $\hat{D}^{-1} = ??$  $\hat{D}^{-1} f = \int_{-\infty}^{\infty} G(x,x') f(x')\, dx'$, "Green's function"

    where $\boxed{\hat{D} G(x,x') = \delta(x-x')}$.

    Then $\hat{D}\hat{D}^{-1} f = \int_{-\infty}^{\infty} \underbrace{\hat{D} G(x,x')}_{\delta(x-x')} f(x')\, dx' = f(x)$

    Therefore,
    $$\boxed{\phi(x) = \int_{-\infty}^{\infty} G(x,x') f(x')\, dx'}.$$