Short Takes 331

Eigensystems
&
Inverses
Part 2

Eigensystems & inverses. Part 2.

Consider A normal (i.e. $A^\dagger A = A A^\dagger$)

→ A is unitarily equivalent to a diagonal matrix D

$$U^\dagger A U = D \quad , \quad U^\dagger = U^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix} \quad \text{matrix of eigenvalues}$$

→ $A = U D U^\dagger$ "spectral decomposition/representation"

$$A_{ij} = \sum_{k=1}^{N} (\vec{v}_k)_i \, \lambda_k \, (\vec{v}_k^*)_j$$

$\vec{v}_k$ eigenvectors

λ_k eigenvalues

→ If A is invertible, we also have

$$(A^{-1})_{ij} = \sum_{k=1}^{N} (\vec{v}_k)_i \, \lambda_k^{-1} \, (\vec{v}_k^*)_j$$

- Note

$$\delta_{ij} = (A^{-1} A)_{ij} = \sum_{q=1}^{N} (A^{-1})_{iq} A_{qj}$$

$$= \sum_{q=1}^{N} \sum_{k=1}^{N} (\vec{v}_k)_i \, \lambda_k^{-1} (\vec{v}_k^*)_q \sum_{k'=1}^{N} (\vec{v}_{k'})_q \, \lambda_{k'} (\vec{v}_{k'}^*)_j$$

$\hookrightarrow \delta_{kk'}$ "orthonormality"

$$= \sum_{k=1}^{N} (\vec{v}_k)_i \, \lambda_k^{-1} \sum_{k'=1}^{N} \lambda_{k'} (\vec{v}_{k'}^*)_j \, \delta_{kk'}$$

$$= \sum_{k=1}^{N} (\vec{v}_k)_i (\vec{v}_k^*)_j = \delta_{ij} \quad \text{"completeness"}$$

?

- Why completeness?

$$\delta_{ij} = \sum_{k=1}^{N} (\vec{V}_k)_i \, (\vec{V}_k^*)_j$$

Take an arbitrary vector $\vec{w}$, then

$$\begin{aligned}
[\mathbb{1} . \vec{w}]_i = (\vec{w})_i &= \sum_j \delta_{ij} (\vec{w})_j = \\
&= \sum_{j,k} (\vec{V}_k)_i (\vec{V}_k^*)_j (\vec{w})_j = \sum_k (\vec{V}_k)_i \, b_k ,
\end{aligned}$$

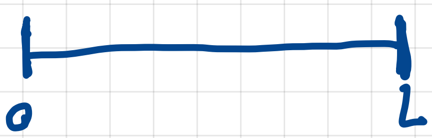
$$b_k = \sum_j (\vec{V}_k^*)_j (\vec{w})_j$$

$$\longrightarrow \boxed{\vec{w} = \sum_k \vec{V}_k \, b_k}$$

i.e. $\vec{w}$ can be written as a linear combination of the $\vec{V}_k$ eigenvectors, with coefficients b_k.

- In the example of the previous video ...

$$A = -\frac{d^2}{dx^2}$$



$$\phi(0) = \phi(L) = 0$$

$$\begin{cases} \psi_k = C \sin\left(\frac{k\pi}{L} x\right) \\ \lambda_k = \left(\frac{k\pi}{L}\right)^2 \end{cases} , \quad k = 1, 2, \ldots$$

orthonormality : $\displaystyle\int_0^L \psi_k^*(x) \, \psi_{k'}(x) \, dx = \delta_{kk'}$

completeness : $\displaystyle\sum_{k=1}^{\infty} \psi_k(x) \, \psi_k^*(x') = \delta(x - x')$