Short Takes 331

Eigensystems & Inverses

Eigensystems & inverses

Recall: given eigenvectors $\vec{v}_k$ and eigenvalues λ_k of a matrix A, we have

$$A \vec{v}_k = \lambda_k \vec{v}_k \quad , \quad k = 1, 2, \dots, N$$

- We can assemble $\vec{v}_k$'s into a matrix:

$$V = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_N \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{pmatrix}$$

and then $AV = VD$, $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix}$

If $\{\vec{v}_k\}$ is a basis, then V^{-1} exists and we can write

$$A = V D V^{-1} \quad \leftarrow \text{spectral representation.}$$

- $\boxed{\det A} = \det(V D V^{-1})$

$$= \det V \det D \underbrace{\det(V^{-1})}_{= (\det V)^{-1}} = \boxed{\det D = \prod_{k=1}^{N} \lambda_k = \lambda_1 \lambda_2 \dots \lambda_N}$$

If $\lambda_k \neq 0$ for all k, then $\det A \neq 0$ and then A^{-1} exists.

- $(M_1 M_2)^{-1} = ?$ $\quad M_1, M_2$ matrices

$$(M_1 M_2)^{-1} = M_2^{-1} M_1^{-1} \rightarrow M_2^{-1} M_1^{-1} M_1 M_2 = \mathbb{1} \ \checkmark$$

Similarly, $(M_1 M_2 \dots M_K)^{-1} = M_K^{-1} M_{K-1}^{-1} \dots M_2^{-1} M_1^{-1}$ for any K.

Then, $$\boxed{A^{-1}} = (V D V^{-1})^{-1} = (V^{-1})^{-1} D^{-1} V^{-1} = \boxed{V D^{-1} V^{-1}}$$

(special case of $f(A) = V f(D) V^{-1}$!)

- Let's take a closer look when $V^{-1} = V^{+}$ (unitary case, A normal)

$$[A^{-1}]_{ij} = \sum_{k=1}^{N} [V]_{ik} \lambda_k^{-1} [V^{+}]_{kj}$$

$$[V]_{ik} = (\vec{v}_k)_i$$

$$[V^{+}]_{kj} = (\vec{v}_k^{*})_j$$

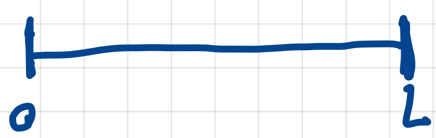
$$= \sum_{k=1}^{N} (\vec{v}_k)_i \lambda_k^{-1} (\vec{v}_k^{*})_j$$

Was also true for A itself:

$$A_{ij} = \sum_{k=1}^{N} (\vec{v}_k)_i \lambda_k (\vec{v}_k^{*})_j$$

- Example

$$A = -\frac{d^2}{dx^2}$$



$\rightarrow A \varphi_k(x) = \lambda_k \varphi_k(x)$ $\qquad \phi(0) = \phi(L) = 0$

$$\begin{cases} \varphi_k = C \sin\left(\frac{k\pi}{L} x\right), & k = 1, 2, \dots \leftarrow \text{eigenvectors} \\ \lambda_k = \left(\frac{k\pi}{L}\right)^2 & \leftarrow \text{eigenvalues} \end{cases}$$

What is the spectral representation? What about A^{-1}?

$$A \overset{?}{\longleftrightarrow} \sum_k \psi_k(x)\, \lambda_k\, \psi_k^*(x')$$

? two variables x, x'??

- How is $A = -\frac{d^2}{dx^2}$ a matrix?

$\rightarrow$ $\underset{\text{op}}{A}\,\underset{\text{vec}}{f} \rightarrow \int A_{x,x'} f(x')\, dx' = \int \delta(x-x')\left(-f''(x')\right) dx' = -f''(x)$

$$A_{x,x'} = \delta(x-x')\left(-\frac{d^2}{dx'^2}\right)$$

Dirac delta
more on this later!

$$A_{x,x'} = \sum_{k=1}^{\infty} \psi_k(x)\, \lambda_k\, \psi_k^*(x')$$

&

$$\boxed{(A^{-1})_{x,x'} = \sum_{k=1}^{\infty} \psi_k(x)\, \lambda_k^{-1}\, \psi_k^*(x')} \leftarrow \text{"Green's function"}$$

Then, when solving $A\phi(x) = f(x) \Rightarrow \phi(x) = A^{-1} f(x)$

$$\rightarrow \boxed{\phi(x)} = \int (A^{-1})_{x,x'}\, f(x')\, dx'$$

$$= \sum_{k=1}^{\infty} \psi_k(x)\, \lambda_k^{-1} \underbrace{\int \psi_k^*(x')\, f(x')\, dx'}_{b_k}$$

$b_k \leftarrow$ calculate for each k, given $f(x)$.

$$\boxed{= \sum_{k=1}^{\infty} \psi_k(x)\, \lambda_k^{-1}\, b_k}$$

Very common approach in boundary-value problems in E&M & beyond!